Modular Curves

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Modular curves are among the most interesting objects on the interface of algebraic geometry and number theory. They are essential in proving many of the modern results about elliptic curves, one of them being the modularity theorem (previously known as Taniyama–Shimura conjecture, relevant also in the proof of Fermat's last theorem). In this workshop, we will see how modular curves are used in showing that any elliptic curve defined over the rational numbers cannot have rational 11-torsion. It is a special case of a theorem of Mazur [2], which makes precise the structure of the rational torsion points on an elliptic curve. The program is roughly built around Weston's introductory survey article [4]. For more details, one can refer to [3], [1].

Talk 1: Setting up

- Introduce congruence subgroups $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$ and mention that the groups $\Gamma(N)$ are torsion free for $N \geq 3$ (possibly prove this).
- Introduce action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear/Möbius transformations on the upper half plane. (possibly show that this is a well-defined action)
- Mention discreteness of the action.
- Show that the modular group is generated by 2 elements.
- Define modular forms and functions, in particular j-invariant and its version(s) for congruence subgroups (in particular j_{11}). Present Eisenstein series as example of modular form and define cusp forms. You can define j-invariant via Eisenstein forms.
- Introduce fundamental domain for modular group and also show how to get fundamental domains for congruence subgroups.

References: [1, 1.1, 1.2, 2.5], [3, I.4] , [4, section 1].

Talk 2: Modular curves as Riemann surfaces

- Define modular curves $Y_0(N)$, $Y_1(N)$ and their respective compactifications $X_0(N)$ and $X_1(N)$. Spend some time to explain how cusps are added to get $X(1)$ and similarly for $X_i(N)$ for $i = 0, 1$.
- Sketch the proof that the modular curve $X(1)$ is a Riemann surface. As another example, show $X_0(11)$ is a complex torus as mentioned in [4, Section 2, p. 6].
- Roughly sketch the proof of Theorem 7.5 of [3]. Use section 3 of [4] as an example.

References: [1, 2.1, 2.2], [3, I.2, II.6, II.7].

Talk 3: Modular curves as moduli spaces

- Recall definition of moduli problem and coarse and fine moduli space.
- Show that complex elliptic curves correspond to lattices in $\mathbb{C}$.
- Set up correspondence of complex elliptic curve (with level structure resp.) with complex points in $X(1)$ ($X_0(N), X_1(N)$ resp.).
- Mention that $X_1(N)$ is fine moduli space for appropriate level structure. (section 9 of [4]).

References: [4, start of Section 2, Sections 6,7], [3, II.8], [1, 1.3-5]

Talk 4: Main result

- Mention Mordell–Weil theorem and motivate the idea to find $E(\mathbb{Q})$ for elliptic curves defined over $\mathbb{Q}$.
- Relate the problem of finding 11-torsion rational point on an elliptic curve $E/\mathbb{Q}$ to computing $X_1(11)(\mathbb{Q})$.

- Do calculations mentioned in [4, Section 4,5,8] to find equation of $X_0(11)$ and mention that similar technique can be used to get equation for $X_1(11)$ and mention its equation [4]. Conclude that $X_1(11)$ has no non-cusp rational point so there is no rational elliptic curve having a rational 11-torsion point.
- Since $X_1(11)(\mathbb{Q})$ is an elliptic curve, we can use Nagell–Lutz theorem (mention only) to calculate its torsion part and calculating rank is already a difficult problem related to Birch–Swinnerton-Dyer conjecture.

References

[1] Fred Diamond and Jerry Michael Shurman. *A first course in modular forms*, volume 228. Springer, 2005.

[2] Barry Mazur. Modular curves and the Eisenstein ideal. *Publications Mathématiques de l'IHÉS*, 47:33–186, 1977.

[3] James S Milne. *Modular functions and modular forms.* 1990. Course Notes of the University of Michigan.

[4] Tom Weston. Modular curves $X_0(11)$ and $X_1(11)$. `https://swc-math.github.io/notes/files/01Weston1.pdf`.