GRK Workshop, *Ci*-Fields

Ultraproducts and transfer principles I

Zeynep Kısakürek

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Ultrafilter, *non-principal*

I: an infinite set, $\wp(I)$: the power set of I

An **ultrafilter** on I is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that

- $I \in \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. In particular,
  - $\emptyset \notin \mathcal{F}$
  - $A \in \mathcal{F}$ and $A \subseteq B \subseteq I \Rightarrow B \in \mathcal{F}$

Remark (Literature-wise)

Any proper collection of elements of $\wp(I)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

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The language of rings:

$$\mathcal{L}_{Ring} = \{+, -, \cdot, 0, 1\}$$

- two binary function symbols $\mathcal{L}_{Ring} = \{+, -, \cdot, 0, 1\}$
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Definition (quite informal)

A **language** $\mathcal{L}$ is a set of function, relation and constant symbols.

An **$\mathcal{L}$-structure** can be defined as a triple $(M, \mathcal{L}, I)$ consisting of a non-empty domain M, language $\mathcal{L}$ and an interpretation function I.

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Ultraproduct Construction via Ultrafilters

Setting

I: an infinite index set with an ultrafilter $\mathcal{F}$ on it

$(\mathcal{M}_i)_{i \in I}$: a family of $\mathcal{L}$-structures

$\mathcal{L} = \mathcal{L}_{Ring} = \{+, -, \cdot, 0, 1\}$
$\rightsquigarrow (\mathcal{M}_i)_{i \in I}$:family of rings

$\mathcal{L} = \mathcal{L}_{ag} = \{+, -, 0\}$
$\rightsquigarrow (\mathcal{M}_i)_{i \in I}$:family of abelian gps

Definition

Consider the Cartesian product $\prod M_i$ as the set of choice functions

$$\{g : I \to \cup M_i : \forall i \in I, g(i) \in M_i\}$$

Define $\sim_{\mathcal{F}}$ on $\prod M_i$ by

$$g \sim_{\mathcal{F}} h \Leftrightarrow \{i \in I : g(i) = h(i)\} \in \mathcal{F}$$

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With $\prod M_i$, $\mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,

The **ultraproduct** $\mathcal{M} = \prod \mathcal{M}_i / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows

- The domain $\mathcal{L} = \prod M_i / \sim_{\mathcal{F}}$ is the set of equivalence classes of $\sim_{\mathcal{F}}$ in $\prod M_i$, denote the eq. cl. by $[g]$ or $[g(i) : i \in I]$
- $\forall$ function symbol $f \in \mathcal{L}$, define $f^{\mathcal{M}}$ by

$$f^{\mathcal{M}}([g_1], \ldots, [g_n]) = [f^{\mathcal{M}_i}(g_1(i), \ldots, g_n(i)) : i \in I]$$

- $\forall$ relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{M}}$ by

$$([g_1], \ldots, [g_n]) \in R^{\mathcal{M}}$$
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$$\{i \in I : ([g_1(i)], \ldots, [g_n(i)]) \in R^{\mathcal{M}_i}\} \in \mathcal{F}$$

- $\forall$ constant symbol $c \in \mathcal{L}$, define $c^{\mathcal{M}}$ by $c^{\mathcal{M}} = [c^{\mathcal{M}_i} : i \in I]$

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An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{or} = \{+, -, \cdot, 0, 1, <\} = \mathcal{L}_{Ring} \cup \{<\}$

Setting:

$\{\mathbb{R} : i \in \mathbb{N}\}$: a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{or}$-structure

$\mathcal{F}$: an ultrafilter on $\mathbb{N}$

$$\rightsquigarrow \mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$$

$$= \{[g] : g(i) \in \mathbb{R}^{\mathbb{N}}\}$$

function symbols

$[g(i) : i \in I] + [h(i) : i \in I] = [g(i) + h(i) : i \in I]$
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relation symbol

$[g] < [h] \Leftrightarrow \{i \in \mathbb{N} : g(i) < h(i)\} \in \mathcal{F}$

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Łoś' theorem - Fundamental Theorem of Ultraproducts

Setting:

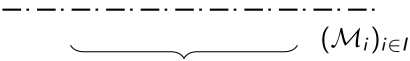
$(\mathcal{M}_i)_{i \in I}$: a family of $\mathcal{L}$-structures

$\mathcal{F}$: an ultrafilter $\mathcal{F}$ on I

$\varphi(\bar{x})$: first order formula in the free variables $\bar{x}$

φ true in $\prod \mathcal{M}_i / \sim_{\mathcal{F}}$

iff



φ true on a "large" subfamily

Theorem (Jerzy Łoś, '55)

For a tuple $([g_1], \ldots, [g_n])$ of elements from $\prod \mathcal{M}_i / \sim_{\mathcal{F}}$,

$$\prod \mathcal{M}_i / \sim_{\mathcal{F}} \models \varphi([g_1], \ldots, [g_n])$$

iff

$$\{i \in I : \mathcal{M}_i \models \varphi(g_1(i), \ldots, g_n(i))\} \in \mathcal{F}$$

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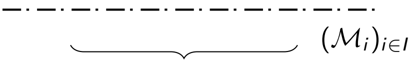

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Previously on this talk...

$\mathcal{L} = \mathcal{L}_{Ring} = \{+, -, \cdot, 0, 1\}$
$\rightsquigarrow (\mathcal{M}_i)_{i \in I}$:family of rings

$\mathcal{L} = \mathcal{L}_{ag} = \{+, -, 0\}$
$\rightsquigarrow (\mathcal{M}_i)_{i \in I}$:family of abelian gps

Definition

With $\prod M_i$, $\mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,

The **ultraproduct** $\mathcal{M} = \prod \mathcal{M}_i / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure

$\rightsquigarrow \mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
$= \{[g] : g(i) \in \mathbb{R}^{\mathbb{N}}\}$

function symbols

$[g(i) : i \in I] + [h(i) : i \in I] = [g(i) + h(i) : i \in I]$
$[g(i) : i \in I].[h(i) : i \in I] = [g(i).h(i) : i \in I]$

relation symbol

$[g] < [h] \Leftrightarrow \{i \in \mathbb{N} : g(i) < h(i)\} \in \mathcal{F}$

constant symbols

zero $[\{0, 0, 0, \ldots\}]$
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$(\mathcal{M}_i)_{i \in I}$:family of rings
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$\mathcal{M} = \prod \mathcal{M}_i / \sim_{\mathcal{F}}$; an $\mathcal{L}_{Ring}$-structure

Is $\mathcal{M}$ a ring?
Answer: YES

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Is $\mathcal{M}$ a ring?
Answer: YES

Corollary

The ultraproduct of groups/rings/fields is again a group/ring/field.

Proposition

If almost all of the K_i are algebraically closed fields, then so is $\prod_{i\in I} K_i/\sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$.

$$(\forall a_0, a_1, \ldots, a_n)(\exists x)(a_n = 0 \vee a_0 + a_1 x + \ldots + a_n x^n = 0)$$

holds for almost all of K_i

$\stackrel{Los}{\Longrightarrow}$ holds for $\prod_{i\in I} K_i/\sim_{\mathcal{F}}$

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Proposition

$\{K_i\}_{i\in I}$: a collection of fields such that for each prime p, only finitely many K_i have characteristic p.

Then $\prod_{i\in I} K_i/\sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$, has characteristic zero.

Consider, for a fixed prime p, $(\exists a)(pa - 1 = 0)$

$$\{i \in I : \text{the statement holds in } K_i\} \in \mathcal{F}$$

$$\stackrel{Los}{\Longrightarrow} \text{the statement holds over } \prod_{i\in I} K_i/\sim_{\mathcal{F}}$$

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Field of Complex Numbers - External Point of View

Setting:

$\mathbb{P} = \{p \in \mathbb{N} : p \text{ prime}\}$
$\{\mathbb{F}_p{}^{alg}\}_{p\in\mathbb{P}}$, as $\mathcal{L}_{Ring}$-structure

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$

$\rightsquigarrow \mathbb{F}^* = \prod_{p\in\mathbb{P}} \mathbb{F}_p{}^{alg} / \sim_{\mathcal{F}}$ is a field

Moreover $\mathbb{F}^*$

- has characteristic 0.
- is algebraically closed.
- has the cardinality of continuum.

$\mathbb{F}^* \simeq \mathbb{C}$

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Non-standard Reals - Internal Point of View

Setting:

$\{\mathbb{R} : i \in \mathbb{N}\}$: a collection of copies of $\mathbb{R}$, as an $\mathcal{L}_{or}$-structure

$\mathcal{F}$: an ultrafilter on $\mathbb{N}$

Consider $\mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$

Observations:

- This structure $\mathcal{R}$ contains infinitesimal numbers.

Consider the eq. cl. $\varepsilon = [\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}]$

$\rightsquigarrow \mathcal{R} \models 0 < \varepsilon$
as $\{n \in \mathbb{N} : 0 < \frac{1}{n}\} = \mathbb{N} \in \mathcal{F}$

Moreover $\mathcal{R} \models \varepsilon < [\{r, r, r, \ldots\}]$, where $r \in \mathbb{R}^{>0}$

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The ultraring $\mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$

Observations:

- $\mathcal{R}$ contains elements larger than any real number

Consider the eq. cl. $\omega = [\{1, 2, 3, \ldots\}]$

$\rightsquigarrow \mathcal{R} \models [\{r, r, r, \ldots\}] < \omega$, for any real number r.

as $\{n \in \mathbb{N} : r < n\} \in \mathcal{F}$

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Consider $(\exists x)(\forall y) y < x$

It does not hold in $\mathbb{R}$, so must be false in $\mathcal{R}$.

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