Can we lift it?

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Yes we can!

Last week

Theorem
Let R be a complete DVR with uniformizing parameter π and $R/\pi^m R$ finite for all $m \geq 1$. Let $(f_1, \dots, f_r) \subset R[T_1, \dots, T_n]$. If

$$\forall m \geq 1 : \exists \underline{a}_m \in (R/\pi^m)^n : \underline{f}(\underline{a}_m) \in \pi^m R,$$

then

$$\exists \underline{a} \in R^n : \underline{f}(\underline{a}) = 0.$$

Can we lift it?
Yes we can!

Road map

This week

See a version of this theorem for *Henselian DVR's*. Use it to prove: if k is C_i, then $k((t))$ is C_{i+1}.

Next week

Arthur gives a proof of the theorem.

Henselian local rings

Definition

A *Henselian local ring* is a local ring $(R, \mathfrak{m}, k)$ such that for any monic $f \in R[T]$ and simple root $a_0 \in k$ of $\bar{f}$, there exists an $a \in R$ such that $f(a) = 0$ and $\bar{a} = a_0$.

Example

1. a field k
2. $\mathbb{Z}_p$, by Hensel's lemma
3. a local ring R such that $\ker(R \to k)$ is nilpotent
4. $k[[t]]$
5. any *complete local ring* (stay tuned)

Henselian local rings

Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:

1. R is Henselian.
2. For $f, g \in R[T]$, f monic, f' invertible in $R[T]_g$ and a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\mathrm{id}_R} & R \\ \downarrow & \nearrow & \downarrow \\ R[T]_g/(f) & \longrightarrow & k, \end{array}$$

   there exists a unique lift.
3. $R \to k$ has the right lifting property with respect to all étale ring maps $A \to B$.
4. Any finite R-algebra S is a finite product of local rings.

Completion

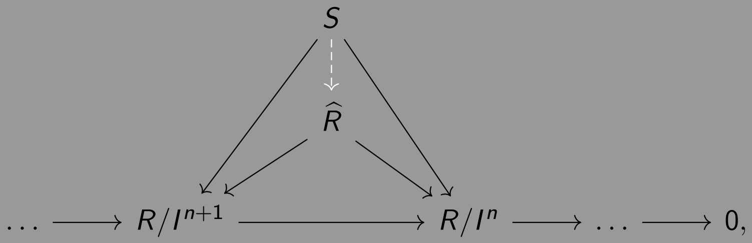
Definition

Let R be a ring and $I \subset R$ an ideal. The *completion of R with respect to I* (or *I-adic completion of R*) is the ring

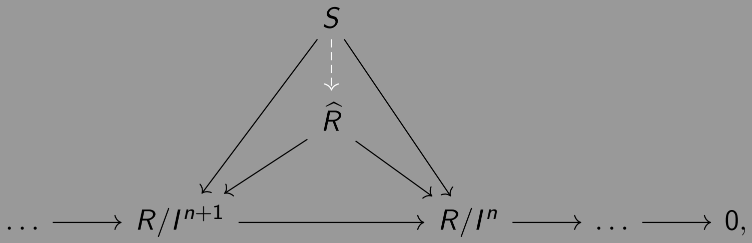
$$\widehat{R}_I = \lim_{n} R/I^n.$$

Call R *complete with respect to I* if $R = \widehat{R}_I$.

Visualize completion as follows:



Completion



Example

1. If I is nilpotent, then $\widehat{R} = R$.
2. If I is idempotent, then $\widehat{R} = R/I$.
3. If $R = k[T]$, then $\widehat{k[T]}_{(T-a)} = k[[T-a]]$.
4. If $R = \mathbb{Z}$, then $\widehat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p$.

Complete local rings are Henselian

Lemma

Let $(R, \mathfrak{m}, k)$ be a complete local ring. Then R is Henselian.

Proof.

Let $f \in R[T]$ monic. Let

1. $f_n \in (R/\mathfrak{m}^{n+1})[T]$ the image of f mod $\mathfrak{m}^{n+1}$
2. f_n' the derivative of f_n with respect to T
3. $a_0 \in k$ a simple root of $f_0(a_0)$.

Assume there exists $a_n \in R/\mathfrak{m}^{n+1}$ such that

$$f_n(a_n) = 0 \quad \text{and} \quad \forall m < n : a_n = a_m \mod \mathfrak{m}^{m+1}.$$

Choose a lift $b \in R/\mathfrak{m}^{n+2}$ of a_n. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$.

Complete local rings are Henselian

Can we lift it?

- $f \in R[T]$
- $f_n, f_n' \in (R/\mathfrak{m}^{n+1})[T]$
- $a_n \in R/\mathfrak{m}^{m+1}$
- $f_n(a_n) = 0$
- $a_n = a_m \mod \mathfrak{m}^{m+1}$
- $b \in R/\mathfrak{m}^{n+2}$
- $b = a_n \mod \mathfrak{m}^{n+1}$
- $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$

Yes we can!

Proof (continued).

Note that $f_{n+1}'(b) = f_0'(a_0) \mod \mathfrak{m}$, so $f_{n+1}'(b)$ is invertible. Set

$$a_{n+1} = b - f_{n+1}(b)/f_{n+1}'(b).$$

Then $a_{n+1} - b \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. May evaluate $f_{n+1}(a_{n+1})$ using Taylor series expansion

$$f_{n+1}(a_{n+1}) = f_{n+1}(b) + (a_{n+1} - b)f_{n+1}'(b).$$

Hence $f_{n+1}(a_{n+1}) = 0$. Get a sequence $a = (a_0, a_1, \dots) \in \lim_n R/\mathfrak{m}^n = R$, such that $f(a) = 0$ and $\bar{a} = a_0$. Thus R is Henselian. □

Greenberg's theorem

The setup

1. Let $(R, \mathfrak{m}, k)$ be a Henselian DVR, $K = \mathrm{Frac}(R)$ its field of fractions, $\widehat{R}$ its completion and $\widehat{K} = \mathrm{Frac}(\widehat{R})$.
2. Let $I = (f_1, \ldots, f_r) \subset R[T_1, \ldots, T_n]$.
3. Assume $K \subset \widehat{K}$ is separable.

Theorem (Greenberg)

There exist $N \geq 1, c \geq 1, s \geq 0$, such that $\forall \nu \geq N$ and diagrams

$$\begin{array}{ccc} R[T_1, \ldots, T_n]/I & \dashrightarrow & R \\ \downarrow & & \downarrow \\ R/\mathfrak{m}^{\nu} & \longrightarrow & R/\mathfrak{m}^{[\nu/c]-s}, \end{array}$$

the answer to the question "Can we lift it?" is "Yes we can!"

From algebra to geometry...

Schemes

- An *affine scheme* is a locally ringed topological space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.
- For $f \in A$, $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ and $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$.
- A *scheme* is a locally ringed topological space $(X, \mathcal{O}_X)$ that is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, "a bunch of rings glued together along localizations."
- For a ring R, the *R-valued points $X(R)$ of X* are maps $\operatorname{Spec} R \to X$.
- $\pi : X \to \operatorname{Spec} R$ is *of finite type* if it is quasi-compact, and $R_f \to \mathcal{O}_X(U)$ is of finite type for every $f \in R$ and open affine $U \subset \pi^{-1}(D(f))$.

From algebra to geometry...

Projective space

- $\mathbb{P}^n_R = \operatorname{Proj} R[T_0, \ldots, T_n]$ can be covered by $n+1$ copies of $\mathbb{A}^n_R = \operatorname{Spec} R[T_0/T_i, \ldots, T_n/T_i] = D_+(T_i)$.
- If $R = k$, then

$$\mathbb{P}^n_k(k) = \{(a_0 : \ldots : a_n) \mid a_i \in k \text{ not all zero}\},$$

  the classical $\mathbb{P}(k^{n+1})$.
- Homogeneous ideal $I \subset R[T_0, \ldots, T_n]$ defines

$$X = \operatorname{Proj} R[T_0, \ldots, T_n]/I \subset \mathbb{P}^n_R,$$

  a closed subscheme. $X(k)$ given by simultaneous roots of $f \in I$.

From algebra to geometry...

- R Henselian DVR
- $N \geq 1$, $c \geq 1$, $s \geq 0$
- $\nu \geq N$
- $X \to \operatorname{Spec} R$ finite type

$$\begin{array}{ccc} R[T_1, \ldots, T_n]/I & \dashrightarrow & R \\ \downarrow & & \downarrow \\ R/\mathfrak{m}^\nu & \longrightarrow & R/\mathfrak{m}^{[\nu/c]-s} \end{array}$$

Corollary (1)

Let $X \to \operatorname{Spec} R$ finite type. Then there exist $N \geq 1$, $c \geq 1$, $s \geq 0$, such that for any $\nu \geq N$ and any diagram

$$\begin{array}{ccc} X & \dashleftarrow & \operatorname{Spec} R \\ \uparrow & & \uparrow \\ \operatorname{Spec} R/\mathfrak{m}^\nu & \longleftarrow & \operatorname{Spec} R/\mathfrak{m}^{[\nu/c]-s}, \end{array}$$

there exists a lift that makes the square commute.

From algebra to geometry...

- R Henselian DVR
- $N \geq 1$, $c \geq 1$, $s \geq 0$
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- $X \to \operatorname{Spec} R$ finite type

$$\begin{array}{ccc} R[T_1, \ldots, T_n]/I & \dashrightarrow & R \\ \downarrow & & \downarrow \\ R/\mathfrak{m}^\nu & \longrightarrow & R/\mathfrak{m}^{[\nu/c]-s} \end{array}$$

Proof.

Let $\{X_i\}_{i \in I}$ a finite affine cover of X. For S a local R-algebra,

$$X(S) = X_i(S).$$

Hence $\operatorname{Spec} R/\mathfrak{m}^\nu \to X$ factors through X_i for some i. Each X_i satisfies Greenberg's theorem with N_i, c_i and s_i. Then $N = \max N_i$, $c = \max c_i$ and $s = \max s_i$ do the job. □

From algebra to geometry...

- R Henselian DVR
- $N \geq 1$, $c \geq 1$, $s \geq 0$
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- $X \to \operatorname{Spec} R$ finite type

$$\begin{array}{ccc} R[T_1, \ldots, T_n]/I & \dashrightarrow & R \\ \downarrow & & \downarrow \\ R/\mathfrak{m}^{\nu} & \longrightarrow & R/\mathfrak{m}^{[\nu/c]-s} \end{array}$$

Corollary (2)

The following are equivalent:

1. $X(R) \neq \emptyset$
2. *for all* $\nu \geq 1$, $X(R/\mathfrak{m}^{\nu}) \neq \emptyset$
3. $X(\widehat{R}) \neq \emptyset$.

Proof.

(1 $\Rightarrow$ 3) is easy. (3 $\Rightarrow$ 2) is easy. (2 $\Rightarrow$ 1) is corollary (1). □

...and from geometry to algebra!

Definition

A domain R is C_i if every homogeneous $f \in R[T_1, \dots, T_n]_d$ of degree d, $n > d^i$, has a nontrivial zero in R.

Lemma

Let R be a C_i-PID and $I \subset R$. Let $f \in (R/I)[T_1, \dots, T_n]_d$, $n > d^i$. Then $\operatorname{Proj}(R/I)[T_1, \dots, T_n]/(f)$ has an (R/I)-valued point.

Proof.

Choose homogeneous $g \in R[T_1, \dots, T_n]_d$ lying over f. Let $S = \operatorname{Proj} R[T_1, \dots, T_n]/(g)$ and $S' = \operatorname{Proj}(R/I)[T_1, \dots, T_n]/(f)$. As R is PID, $S(R) \neq \emptyset$ if and only if g has a nontrivial zero in R. Thus $S(R) \neq \emptyset$ by assumption, so $S(R/I) \neq \emptyset$, so $S'(R/I) \neq \emptyset$. □

...and from geometry to algebra!

Theorem (Greenberg)

Let k be a C_i-field. Then $k((t))$ is C_{i+1}.

Proof.

It suffices to show that $R = k[[t]]$ is C_{i+1}. Let $f \in R[T_1, \ldots, T_n]_d$, $n > d^i$. Set $X = \operatorname{Proj} R[T_1, \ldots, T_n]/(f)$. Then X is of finite type over R. Fix $\nu \geq 1$. There is a map

$$(R/t^\nu)[T_1, \ldots, T_n]/(f_{<\nu}) \longleftarrow R[T_1, \ldots, T_n]/(f)$$
$$X' \longrightarrow X.$$

Note that $R/t^\nu = k[t]/t^\nu$. As $k[t]$ is C_{i+1}, the lemma above gives $x \in X'(R/t^\nu)$. Hence $X(R/t^\nu) \neq \emptyset$, so by corollary (2), $X(R) \neq \emptyset$. Thus $k[[t]]$ is C_{i+1}. $\square$

Can we lift it?
Yes we can!

Questions?