Extensions of C_i fields

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November 6, 2020

Overview

Normic Forms

We start by recalling the definition of normic forms

Definition

A form f of degree d in n variables with coefficients in a field k is said to be *normic of order* i if $n = d^i$ and the only zero of f is the trivial one. When $i = 1$ the form is simply called *normic*.

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In the rest of the talk we will only be concerned with normic forms, i.e. of order 1.

Example

Over the field $\mathbb{Q}$ the form

$$f(x, y) = x^2 + y^2$$

is normic of degree 2.

Why the Name "Normic" Forms?

Lemma A

Let E/k be a finite field extension of degree $e > 1$, then the norm of the extension, $N := N_{E/k}$ is a normic form of degree e.

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Proof.

We fix a basis of E as a k vector space. Then $N(x)$ becomes a homogeniuous polynomial of degree e in the coefficients of x, and we know from field theory that $N(x) = 0 \iff x = 0$, so N is normic. □

Normic Forms of Arbitrarily High Degree

Lemma B

Let k be a field. If k is not algebracially closed, then k admits a normic form of arbitrarily high degree.

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Since k is not algebraically closed, we can find some normic form over k. For instance, we can find a finite extension of k and take it's norm. So let ϕ be such a normic form, and denote by e the degree of ϕ.

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We define the following iterations of ϕ:

$$\begin{aligned} \phi^{(1)} &= \phi(\phi|\phi|\dots|\phi), \\ \phi^{(2)} &= \phi^{(1)}(\phi|\phi|\dots|\phi), \\ &\vdots \end{aligned}$$

Normic Forms of Arbitrarily High Degree

Proof continued.

These iterations are defined as follows: To define $\phi^{(1)}$, we substitute ϕ in for each of the variables in ϕ, and the vertical line is meant to indicate that each ϕ takes a new set of variables. Therefore, since ϕ has degree e (and is a form in e variables since it is normic) we see that $\phi^{(1)}$ is a form of degree e^2 in e^2 variables. In general $\phi^{(m)}$ is a form of degree e^{m+1} in e^{m+1} variables.

Caveat

Greenberg claims that $\phi^{(m)}$ has degree e^m, not e^{m+1} like I claim. Please correct me if I am wrong.

Normic Forms of Arbitrarily High Degree

Example interlude

Consider again the normic form $f(x, y) = x^2 + y^2$ of degree 2 over $\mathbb{Q}$. We have

$$\begin{aligned} f^{(1)}(x, y, z, w) &= f(f|f) \\ &= f(f(x, y), f(z, w)) \\ &= f(x^2 + y^2, z^2 + w^2) \\ &= x^4 + 2x^2y^2 + y^4 + z^4 + 2z^2w^2 + w^4, \end{aligned}$$

a form of degree $4 = 2^2$ over $\mathbb{Q}$.

proof continued

These iterations are defined as follows: To define $\phi^{(1)}$, we substitute ϕ in for each of the variables in ϕ, and the vertical line is meant to indicate that each ϕ takes a new set of variables. Therefore, since ϕ has degree e (and is a form in e variables since it is normic) we see that $\phi^{(1)}$ is a form of degree e^2 in e^2 variables. In general $\phi^{(m)}$ is a form in e^{m+1} in e^{m+1} variables.

Each of these $\phi^{(m)}$ is normic. Consider $\phi^{(1)}$, if $\phi^{(1)}(\underline{x}) = 0$ for some $\underline{x} = (x_1, \dots, x_e, x_{e+1}, \dots, x_{e^2})$, then since $\phi^{(1)} = \phi(\phi| \dots |\phi)$ and ϕ is normic we see that we must have $\underline{x} = 0$, so $\phi^{(1)}$ is normic. The statement for $\phi^{(m)}$ follows by induction. □

Lang-Nagata

Lang-Nagata Theorem

Let K be a C_i field and let $f_1, \ldots, f_r$ be forms in n variables of degree d. If $n > rd^i$ then they have a non-trivial common zero in K.

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Proof

If K is algebraically closed (so $i = 0$), then each f_i defines a hypersurface H_i in $\mathbb{P}_K^{n-1}$. The dimension of the intersection $\bigcap_{1 \leq i \leq r} H_i$ is then greater than or equal to $n - 1 - r \geq 0$ so in particular the f_i's have a common non-trivial zero.

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So we can assume K is not algebraically closed. Then we know by Lemma B that we can find a normic form of degree $e \geq r$, let ϕ be such a form.

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So we can assume K is not algebraically closed. Then we know by Lemma B that we can find a normic form of degree $e \geq r$, let ϕ be such a form.

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$$\begin{aligned}
\phi^{(1)} &= \phi(f_1, \dots, f_r | f_1, \dots, f_r | \dots | f_1, \dots, f_r | 0, \dots, 0), \\
\phi^{(2)} &= \phi^{(1)}(f_1, \dots, f_r | f_1, \dots, f_r | \dots | f_1, \dots, f_r | 0, \dots, 0), \\
&\vdots
\end{aligned}$$

where as before, the vertical lines indicate that we introduce new variables. We fit as many complete sets of f_i into ϕ and fill the rest with zeros.

Lang-Nagata

Example Interlude

If $e = r$ then

$$\phi^{(1)} = \phi(f_1, \ldots, f_r),$$

If $e = 2r + 1$ then

$$\phi^{(1)} = \phi(f_1, \ldots, f_r | f_1, \ldots, f_r | 0),$$

etc.

Lang-Nagata

Proof Continued.

We see that $\phi^{(1)}$ has $n\lfloor \frac{e}{r} \rfloor$ variables and degree de. We have $\lfloor \frac{e}{r} \rfloor \leq \frac{e}{r} < \lfloor \frac{e}{r} \rfloor + 1$, and so

$$de < dr(\lfloor \frac{e}{r} \rfloor + 1).$$

If K is C_1 then we want to have $n\lfloor \frac{e}{r} \rfloor \geq dr(\lfloor \frac{e}{r} \rfloor + 1)$, i.e.

$$(n - dr)\lfloor \frac{e}{r} \rfloor > dr.$$

This we can ensure, since $n - dr > 0$ by assumption, and we can by Lemma B chose e to be arbitrarily large. Since K is C_1, $\phi^{(1)}$ has a non-trivial zero, and that gives us a non-trivial common zero of $f_1, \dots, f_r$ since ϕ is normic.

Lang-Nagata

Proof Continued

Now let K be a C_i field with $i > 1$. We have to analyse $\phi^{(m)}$ for higher m's now. Inductively it is easy to see that the degree of $\phi^{(m)}$ is $d^m e$, and if we denote the number of variables in $\phi^{(m)}$ by N_m then

$$N_{m+1} = n\lfloor \frac{N_m}{r} \rfloor. \qquad (*)$$

Caveat.

Greenberg writes here $N_{m+1} = \lfloor \frac{N_m}{r} \rfloor$, but I am pretty sure the factor of n should be there. Please let me know if I'm mistaken.

Proof Continued

Now let K be a C_i field with $i > 1$. We have to analyse $\phi^{(m)}$ for higher m's now. Inductively it is easy to see that the degree of $\phi^{(m)}$ is $d^m e$, and if we denote the number of variables in $\phi^{(m)}$ by N_m then

$$N_{m+1} = n\lfloor \frac{N_m}{r} \rfloor. \qquad (*)$$

Our aim now is to choose m large enough to ensure that $N_m > (D_m)^i$, where $D_m = d^m e$ denotes the degree of $\phi^{(m)}$. Again, since $\lfloor \frac{N_m}{r} \rfloor \leq \frac{N_m}{r} < \lfloor \frac{N_m}{r} \rfloor + 1$, we can write

$$\lfloor \frac{N_m}{r} \rfloor = \frac{N_m}{r} - \frac{t_m}{r}, \qquad (**)$$

where this remainder term t_m satisfies $0 \leq t_m < r$.

Lang-Nagata

Proof Continued.

We have

$$\begin{aligned}
\frac{N_{m+1}}{D^i_{m+1}} &= \frac{n\lfloor \frac{N_m}{r} \rfloor}{d^i D^i_m} \text{ by definition of degree and } (*) \\
&= \frac{n}{rd^i}\frac{N_m}{D^i_m} - \frac{n}{rd^i}\frac{t_m}{e^i(d^i)^m} \text{ by } (**) \\
&\geq \frac{n}{rd^i}\frac{N_m}{D^i_m} - \frac{n}{rd^i}\frac{r}{e^i(d^i)^m} \text{ since } 0 \leq t_m < r.
\end{aligned}$$

Lang-Nagata

Proof Continued.

We use this same inequality for all $j \leq m$ and obtain

$$\begin{aligned}
\frac{N_{m+1}}{D^i_{m+1}} &\geq \frac{n}{rd^i}\frac{N_m}{D^i_m} - \frac{n}{rd^i}\frac{r}{e^i(d^i)^m} \\
&\geq (\frac{n}{rd^i})^2(\frac{N_{m-1}}{D^i_{m-1}} - \frac{r}{e^i(d^i)^{m-1}}) - (\frac{n}{rd^i})(\frac{r}{e^i(d^i)^m}) \\
&\vdots \\
&\geq (\frac{n}{rd^i})^m\frac{N_1}{D^i_1} - \frac{r}{e^i}\frac{n}{r}\frac{1}{(d^i)^{m+1}}(\sum_{j=0}^{m-1}(\frac{n}{r})^j) \\
&= (\frac{n}{rd^i})^m\frac{N_1}{D^i_1} - \frac{r}{e^i}\frac{n}{r}\frac{1}{(d^i)^{m+1}}\frac{(\frac{n}{r})^m - 1}{\frac{n}{r} - 1}.
\end{aligned}$$

Lang-Nagata

Proof Continued.

We plug in $D_1 = ed$, $N_1 = n\lfloor \frac{e}{r} \rfloor$ and write $\lfloor \frac{e}{r} \rfloor = \frac{e}{r} - \frac{t}{r}$ where $0 \leq t < r$.

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$$\begin{aligned}
\frac{N_{m+1}}{D^i_{m+1}} &\geq (\frac{n}{rd^i})^{m+1}\frac{e-t}{e^i} - \frac{r}{e^i}\frac{n}{r}\frac{1}{(d^i)^{m+1}}\frac{r(n^m - r^m)}{r^m(n-r)} \\
&= (\frac{n}{rd^i})^{m+1}\frac{e-t}{e^i} - \frac{r}{e^i}\frac{n}{rd^i}\frac{n}{n-r}((\frac{n}{rd^i})^m - \frac{1}{(d^i)^m}) \\
&= (\frac{n}{rd^i})^{m+1}(\frac{e-t}{e^i} - \frac{r^2}{e^i(n-r)}) + \frac{1}{(d^i)^m}(\frac{rn}{e^id^i(n-r)}) \\
&= (\frac{n}{rd^i})^{m+1}\frac{(n-r)(e-t) - r^2}{e^i(n-r)} + \frac{1}{(d^i)^m}(\frac{rn}{e^id^i(n-r)}).
\end{aligned}$$

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Proof Continued.

Again we can use Lemma B to choose e as large as we want, so we choose it such that $(n-r)(e-t)-r^2>0$. Since $\frac{n}{rd^i}>1$ (we have $n>rd^i$ by assumption) we see that the first term tends to ∞ as $m\to\infty$. The second term tends to 0 as $m\to\infty$ so we see that $\frac{N_m}{D_m^i}\to\infty$ as $m\to\infty$.

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We can thus find some m such that $N_m>D_m^i$, but then $\phi^{(m)}$ has a non-trivial zero, and that will give us a non-trivial common zero of $f_1,\ldots,f_r$.

□

Extensions.

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Theorem 4

Every algebraic extension of a C_i field is C_i.

Theorem 5

If K is a C_i field and E/K is a an extension of trancendence degree j, then E is a C_{i+j} field.

Theorem 4

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Proof.

Let K be a C_i field. It is enough to prove the statement for finite extensions E/K since the coefficients of any given form over E lie in a finite extension over K.

Theorem 4

Proof continued.

Fix a basis $b_1, \ldots, b_e$ of E as a K-vector space. Let $f(x_1, \ldots, x_n)$ be a form of degree d over E with $n > d^i$.

Theorem 4

Proof continued.

Fix a basis $b_1, \ldots, b_e$ of E as a K-vector space. Let $f(x_1, \ldots, x_n)$ be a form of degree d over E with $n > d^i$.

We introduce new variables y_{ij} with

$$x_i = \sum_{j=1}^{e} y_{ij} b_j.$$

Then

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{e} f_i(\underline{y}) b_i$$

where the f_i are forms in en variables of degree d over K.

Theorem 4

Proof Continued.

Finding a zero for f in E is then equivalent to finding a common zero of $f_1, \ldots, f_e$ in K. Now $en > ed^i$ by the assumption on n and d, and so by the Lang-Nagata theorem we can find a non-trivial zero for f, and E is therefore a C_i field. □

Theorem 5

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If K is a C_i field and E/K is a an extension of trancendence degree j, then E is a C_{i+j} field.

Proof.

E is an algebraic extension of a purely trancendental extension of K. By Theorem 4, we can assume E is purely trancendental over K. Furthermore, we can by induction reduce to the case where $E = K(T)$.

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E is an algebraic extension of a purely trancendental extension of K. By Theorem 4, we can assume E is purely trancendental over K. Furthermore, we can by induction reduce to the case where $E = K(T)$.

We can always clear denominators, so we can reduce to considering forms with coefficients in the polynomial ring $K[T]$.

Theorem 5

Proof Continued.

Suppose $f(x_1, \ldots, x_n)$ is a form of degree d with coefficients in $K[T]$. We introduce new variables y_{ij} with

$$x_i = \sum_{j=0}^{s} y_{ij} T^j.$$

We specify what this s is later.

Theorem 5

Proof Continued.

Suppose $f(x_1, \ldots, x_n)$ is a form of degree d with coefficients in $K[T]$. We introduce new variables y_{ij} with

$$x_i = \sum_{j=0}^{s} y_{ij} T^j.$$

We specify what this s is later.

Let r be the highest degree occurring in a coefficient of f, then we can write

$$f(x_1, \ldots, x_n) = \sum_{j=0}^{ds+r} f_j(\underline{y}) T^j,$$

where each f_j is a form over K of degree d in $n(s+1)$ variables.

Theorem 5

Proof Continued.

We now specify what this s is. It is some positive integer large enough so that we have

$$n(s+1) > d^i(ds + r + 1).$$

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We now specify what this s is. It is some positive integer large enough so that we have

$$n(s+1) > d^i(ds+r+1).$$

We rewrite this as

$$(n-d^{i+1})s > d^i(r+1) - n,$$

and notice that by assumption $n > d^{i+1}$ and the quantity on the right-hand side is fixed, so we can choose such a large enough s.

Proof Continued.

We can now use the Lang-Nagata theorem to find a non-trivial common zero of the $f_0, \dots, f_{ds+r}$, which is precisely the same as finding a non-trivial zero for f, showing that $K(T)$ is a C_{i+1} field.

□

Thank You!