Moduli spaces of vector bundles on curves

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Overview

Aim

- *Define vector bundles, relate them to locally free sheaves*
- *Study line bundles on smooth projective curves*
- *Define Hilbert schemes*
- *Discuss semistability*
- *Construct moduli spaces*
- *Study properties of moduli spaces*

We follow Part I of J. Le Potier: Lectures on Vector Bundles.

Definition of vector bundles

Definition

Let X be a connected $\mathbb{C}$-variety.

- A linear fibration on X is a map $p : E \to X$ of varieties, together with a vector space structure on every fibre $p^{-1}(x) =: E_x$.
- A map of linear fibrations $p : E \to X$, $p' : E' \to X$ is a map $\varphi : E \to E'$ such that $p'\varphi = p$ and $\varphi : E_x \to E'_x$ linear for all $x \in X$.
- V fin.dim. $\mathbb{C}$-vector space: $\mathrm{pr}_1 : X \times V \to X$ trivial fibration.
- A vector bundle on X is a linear fibration which is locally isomorphic to a trivial one. That is: exists open covering $X = \bigcup_i U_i$ such that $U_i \times V \simeq p^{-1}(U_i)$ for all i.
- Its rank $\mathrm{rk}(E) = \dim V$.

One can define direct sums, subbundles, quotient bundles of vector bundles. But in general we don't have kernels, images, cokernels.

Example

$X = \mathbb{A}^1$, $V \neq 0$ vector space

$$\varphi : X \times V \to X \times V, \quad (\lambda, v) \mapsto (\lambda, \lambda v).$$

Kernel is a linear fibration E with $E_0 = V$ and $E_\lambda = 0$ for $\lambda \neq 0$.

The category of vector bundles

Theorem

The category of vector bundles on X is equivalent to the category of locally free (coherent) sheaves on X.

Idea of proof: associate to the vector bundle $p : E \to X$ its sheaf of sections

$$U \mapsto \Gamma(U, E) := \{s : U \to E \mid p \circ s = \mathrm{id}_U\},$$

which is a $\mathcal{O}_X(U)$-module. This sheaf is locally isomorphic to $\mathcal{O}_X \otimes V$.

Line bundles on curves

From now on X is a smooth projective curve. In the $\mathbb{C}$-topology, it is a compact Riemann surface of some genus $g \geq 0$.

Line bundles, that is, vector bundles of rank 1 form group under $\otimes$, the Picard group $\mathrm{Pic}(X)$ of X. It is isomorphic to the group $\mathrm{Div}(X)$ of divisors modulo linear equivalence. We have an exact sequence

$$1 \to \mathrm{Jac}(X) = \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \simeq \mathrm{Div}(X) \stackrel{\deg}{\to} \mathbb{Z} \to 1$$

where $\deg \sum_{x \in X} \lambda_x x = \sum_x \lambda_x$, and $\mathrm{Jac}(X)$ is an abelian variety, topologically isomorphic to $\mathbb{C}^g / \mathbb{Z}^{2g}$.

We can generalize the degree deg to arbitrary rank bundles by $\deg(E) := \deg(\bigwedge^{\mathrm{rk} E} E)$.

Classification of topological bundles

Theorem

Topologically, rank and degree are the only invariants of a vector bundle on X.

Hilbert polynomials of coherent sheaves

The given embedding $X \subset \mathbb{P}^n$ induces line bundle $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be a coherent sheaf on X (it is then isomorphic to the direct sum of a locally free sheaf and a torsion sheaf, concentrated on finitely many points of X). Has coherent cohomology $H^i(X, \mathcal{F})$ for $i = 0, 1$, Euler characteristic $\chi(\mathcal{F}) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F})$.
Twist of coherent sheaf: $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$.

Fact

Exists Hilbert polynomial $P(t) \in \mathbb{Z}[t]$ such that $P(n) = \chi(\mathcal{F}(n))$ for all n. Namely, $P(t) = \mathrm{rk}(\mathcal{F}) \deg(\mathcal{O}_X(1))t + \chi(\mathcal{F})$.

Hilbert schemes

Theorem

Take $\mathcal{E}$ locally free, P linear polynomial. Exists projective variety $\mathrm{Hilb}^P(\mathcal{E})$ parametrizing quotients of $\mathcal{E}$ with Hilbert polynomial P.

Sketch of proof: One can choose N large enough such that the following holds:

given a quotient $\mathcal{E}/\mathcal{F}$, the subspace $H^0(X, \mathcal{F}(N)) \subset H^0(X, \mathcal{E}(N))$ has codimension $P(N)$ and determines $\mathcal{E}/\mathcal{F}$ as cokernel of $H^0(X, \mathcal{F}(N)) \otimes \mathcal{O}_X(-N) \to \mathcal{E}$.

In this way, we realize $\mathrm{Hilb}^P(\mathcal{E})$ as closed subvariety of Grassmannian $\mathrm{Gr}^{P(N)}(H^0(X, \mathcal{E}(N)))$.

(Semi-)stability

Definition

- Slope of $0 \neq E$ is $\mu(E) = \deg(E)/\mathrm{rk}(E)$.
- E is called semistable if $\mu(F) \leq \mu(E)$ for all subbundles $0 \neq F \subset E$.
- E is called stable if $\mu(F) < \mu(E)$ for all proper subbundles $0 \neq F \subset E$.

Fact

- *Semistable bundles of a fixed slope form an abelian finite length category.*
- *In particular, its simple/irreducible objects are the stables of that slope, and any semistable admits Jordan-Hölder type filtration by stables of same slope.*
- *An arbitrary vector bundle admits a unique Harder-Narasimhan filtration: a filtration with semistable subquotients of decreasing slope.*

Reminder on GIT and quotients

Let G be a linearly reductive group, acting linearly on some vector space Z, and assume $Y \subset \mathbb{P}(Z)$ G-stable closed subvariety. Homogeneous coordinate ring $\mathbb{C}[Y] = \mathbb{C}[Z]/\mathbb{V}(Y)$ contains (graded, finitely generated) ring of G-invariants $\mathbb{C}[Y]^G$.

Definition

$y \in Y$ semistable if $f(y) \neq 0$ for some $0 \neq f \in \mathbb{C}[Y]^G$ of positive degree.

Theorem

Then $Y^{\rm sst} \to Y^{\rm sst}//G := \mathrm{Proj}(\mathbb{C}[Y]^G)$ *is a categorical quotient: it is surjective and every fibre contains a unique closed* G*-orbit in* $Y^{\rm sst}$*. That is:* $Y^{\rm sst}//G$ *parametrizes semistable closed* G*-orbits in* Y*.*

An example of semistability

Semistability can be verified numerically using the Hilbert-Mumford criterion. We will only see this in an example:

$\mathrm{SL}(V)$ acts linearly on V, thus on any $V \otimes W$, thus on any $\bigwedge^p (V \otimes W)$, thus on

$$\mathrm{Gr}^p(V \otimes W) \subset \mathbb{P}(\bigwedge^p (V \otimes W)).$$

Then $U \subset V \otimes W$ defines a semistable point iff for all proper non-zero $V' \subset V$, we have

$$\frac{\dim U \cap (V' \otimes W)}{\dim V'} \leq \frac{\dim U}{\dim V}.$$

Construction of moduli spaces of vector bundles

Finally we can construct a moduli space for vector bundles of rank $r \in \mathbb{N}$ and degree $d \in \mathbb{Z}$ on X, more precisely a variety whose points parametrize semistable such vector bundles up to Jordan-Hölder equivalence (we can't do better).

Tensoring with a degree 1 line bundle induces bijection between isoclasses of (semistable) bundles of rank r, degree d and bundles of rank r, degree $d + r$. We can thus choose wlog d arbitrarily large. In fact – after many subtle and delicate numerical considerations which we have to omit completely – so large that:

every rank r degree d bundle is then a quotient of $\mathcal{O}_X \otimes V$ (for a certain V of dimension $\chi = d + r(1 - g)$) with Hilbert polynomial $rt + \chi$.

every rank r degree d bundle is then a quotient of $\mathcal{O}_X \otimes V$ (for a certain V of dimension $\chi = d + r(1-g)$) with Hilbert polynomial $rt + \chi$.
And (for N large enough)

$$\mathrm{Hilb}^{rt+\chi}(\mathcal{O}_X \otimes V) \subset \mathrm{Gr}^{rN+\chi}(V \otimes H^0(X, O_X(N))).$$

A bundle is semistable (by Hilbert-Mumford criterion) iff the corresponding subspace U in the Grassmannian is semistable for the $\mathrm{SL}(V)$-action. Two bundles are isomorphic iff the corresponding subspaces in the Grassmannian are conjugate under $\mathrm{SL}(V)$. Thus we finally define:

Definition

$$M(r,d) := \mathrm{Hilb}^{rt+\chi}(\mathcal{O}_X \otimes V)^{\mathrm{sst}} // \mathrm{SL}(V).$$

Properties of the moduli spaces

Theorem

Assume $g \geq 2$.

- $M(r,d)$ *parametrizes semistable rank r degree d bundles on X up to Jordan-Hölder equivalence.*
- *Exists open subset $M^s(r,d)$ parametrizing stable bundles up to isomorphism.*
- $M^s(r,d) \neq \emptyset$ *for all r and d.*
- $M(r,d)$ *is irreducible of dimension $(g-1)r^2+1$.*
- $M(r,d)$ *is smooth if $\gcd(r,d)=1$.*

Final slide

Thank you!