Midsummmer resolutions

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Derived categories, stable module categories and cohomology in practice

Derived categories

Construction of a derived category

1. Take your favorite abelian category $\mathcal{A}$.
2. Form the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$.
3. Define chain homotopies between chain maps.
4. Consider maps up to chain homotopy to obtain the homotopy category $K(\mathcal{A})$.
5. Define quasi-isomorphisms as chain maps that induce isomorphisms in (co)homology.
6. Localize with respect to quasi-isomorphisms to obtain the derived category $D(\mathcal{A})$.

Derived categories are triangulated; they come equipped with a class of distinguished triangles that act as exact sequences and are rotatable.

Short exact sequences are distinguished triangles

Let $\mathcal{A}$ be an abelian category and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

a short exact sequence in $\mathcal{A}$. Note that $C(f)$ in $\mathrm{Ch}(\mathcal{A})$ is the complex $0 \to A \to B \to 0$ with B in degree 0. The morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism since its cone is a short exact sequence and therefore acyclic.

Short exact sequences are distinguished triangles

Therefore, in $D(\mathcal{A})$, there is an isomorphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C(f) & \longrightarrow & A[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle -1} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & A[1], \end{array}$$

so the short exact sequence defines a distinguished triangle.

Sheaf cohomology

Let X be a scheme, $\mathcal{A}$ the abelian category of quasi-coherent $\mathcal{O}_X$-modules, $\Gamma(X,-) : \mathcal{A} \to \mathcal{A}$ the left exact global sections functor and $\mathcal{F} \in \mathcal{A}$. The sheaf cohomology $H^\bullet(X, \mathcal{F})$ is the cohomology

$$H^\bullet(\mathbb{R}\Gamma(X, \mathcal{F}))$$

of the right derived functor $\mathbb{R}\Gamma(X,-) : D(\mathcal{A}) \to D(\mathcal{A})$. Recipe:

1. Take an injective resolution $\mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \dots$, or equivalently, an isomorphism $\mathcal{F} \to \mathcal{I}_\bullet$ in $D(X)$ with $\mathcal{I}_\bullet$ $\Gamma(X,-)$-acyclic.
2. Apply $\Gamma(X,-)$ to $\mathcal{I}_\bullet$ to obtain $\mathcal{I}_0(X) \to \mathcal{I}_1(X) \to \dots$.
3. Take the cohomology of $\mathcal{I}_\bullet(X)$.

Problem: injective resolutions are annoying.
Solution: Čech cohomology! (But we won't go into this.)

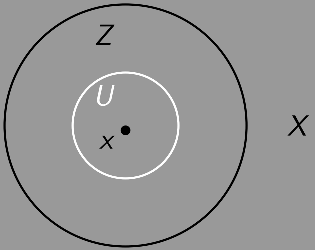
Serre's criterion for affineness

Theorem

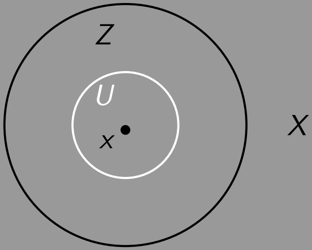
Let X be a quasi-compact scheme. Assume that, for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, $H^1(X, \mathcal{I}) = 0$. Then X is affine.

Proof.

$x \in X$ closed point, $x \in U \subset X$ affine open neighborhood,
$Z = X \setminus U$, $Z' = Z \cup \{x\}$, $\mathcal{I}$ and $\mathcal{I}'$ corresponding ideals.



Serre's criterion for affineness



$$
\begin{aligned}
U &= \operatorname{Spec} A \\
x &= \mathfrak{m} \subset A
\end{aligned}
$$

There are exact sequences

$$0 \longrightarrow \mathcal{I}' \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}' \longrightarrow 0$$

$$0 \longrightarrow H^0(X, \mathcal{I}') \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{I}/\mathcal{I}') \longrightarrow H^1(X, \mathcal{I}'),$$

and $H^1(X, \mathcal{I}') = 0$ by assumption. Note that $H^0(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$, so there exists $f \in \mathcal{I}(X)$ mapping to $1 \in A/\mathfrak{m}$, so $X_f \subset U$ is affine.

Serre's criterion for affineness

Let $I = \{f \in \mathcal{O}_X(X) \mid X_f \text{ affine}\}$ and $W = \bigcup_{f \in I} X_f$. Then $X \setminus W$ is quasi-compact, hence contains a closed point if it is non-empty, hence is empty.

Choose finitely many $f_1, \ldots, f_n \in I$ such that

$$X = X_{f_1} \cup \cdots \cup X_{f_n}.$$

Then there is a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Let $\mathcal{F}_i$ be the first i summands of $\mathcal{F}$. The exact sequence

$$H^1(X, \mathcal{F}_1) \longrightarrow H^1(X, \mathcal{F}_2) \longrightarrow H^1(X, \mathcal{F}_2/\mathcal{F}_1)$$

begins and ends with 0. Hence $H^1(X, \mathcal{F}_2) = 0$, and $H^1(X, \mathcal{F}) = 0$ by iteration. Therefore, $\mathcal{O}_X(X)^{\oplus n} \to \mathcal{O}_X(X)$ is surjective. $\square$

Tor

Let R be a commutative ring, $\mathcal{A}$ the abelian category of R-modules and A an R-module. Let $F : \mathcal{A} \to \mathcal{A}$ be the right exact functor $A \otimes -$. For $B \in \mathcal{A}$,

$$\mathrm{Tor}_i^R(A, B) = H_i(A \otimes^{\mathbb{L}} B),$$

where $A \otimes^{\mathbb{L}} - : D(R) \to D(R)$ is the left derived functor of F.
Recipe:

1. Take a projective resolution $\ldots \to P_1 \to P_0 \to B$, or equivalently, an isomorphism $P_\bullet \to B$ in $D(R)$ with $P_\bullet$ F-acyclic.
2. Tensor $P_\bullet$ with A to get $B'_\bullet : \ldots \to A \otimes P_1 \to A \otimes P_0 \to 0$.
3. Compute $H_i(B'_\bullet)$.

Note that $\mathrm{Tor}_0^R(A, B) = A \otimes B$.

Tor

Properties

1. $A \in \mathcal{A}$ is flat if and only if $\mathrm{Tor}_n^R(A, B) = 0$ for all $B \in \mathcal{A}$ and $n \geq 1$.
2. For a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that C is flat, A is flat if and only if B is flat.
3. $\mathrm{Tor}_n^R(A, -)$ and $\mathrm{Tor}_n^R(-, B)$ can be computed using flat resolutions.
4. $\mathrm{Tor}_n^R(A, -)$ commutes with colimits. In particular, it commutes with direct sums.
5. For the following items, assume R is a domain. Let $Q = \mathrm{Frac}(R)$ and $K = Q/R$. Then $\mathrm{Tor}_1^R(K, -)$ is naturally isomorphic to the torsion functor $\mathrm{Tor} : \mathcal{A} \to \mathcal{A}$.
6. $\mathrm{Tor}_n^R(A, B)$ is torsion for all $n \geq 1$.

Tor

Example

Let k a field, $R = k[x, y]$ and $I = (x, y)$. Then $x \otimes y - y \otimes x \neq 0$ in $I \otimes I$. Consider the exact sequence

$$\mathrm{Tor}_1^R(k, I) \longrightarrow I \otimes_R I \longrightarrow R \otimes_R I \longrightarrow k \otimes_R I \longrightarrow 0.$$

Note that R is a domain, so I is a torsion-free R-module. But, the second map is not injective since $x \otimes y - y \otimes x \mapsto 0$, so $\mathrm{Tor}_1^R(k, I) \neq 0$, so I is not a flat R-module!

Ext

Let $G : \mathcal{A} \to \mathcal{A}$ be the left exact functor $\mathrm{Hom}(A, -)$. For $B \in \mathcal{A}$,

$$\mathrm{Ext}^i_R(A, B) = H^i(\mathbb{R}\mathrm{Hom}(A, B)) = D(\mathcal{A})(A, B[i]),$$

where $\mathbb{R}\mathrm{Hom}(A, -)$ is the right derived functor of G. Then $\mathbb{R}\mathrm{Hom}(A, -)$ and $A \otimes^{\mathbb{L}} -$ form an adjoint pair of functors $\mathcal{A} \to \mathcal{A}$.
Recipe:

1. Take an injective resolution $I_\bullet$ of B.
2. Compute the complex $\mathrm{Hom}(A, I_\bullet)$.
3. Take the cohomology of the complex.

Ext

Let

$$0 \longrightarrow B \longrightarrow C_1 \longrightarrow \ldots \longrightarrow C_i \longrightarrow A \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$.Then

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & A \\
\downarrow & & & & \downarrow & & \downarrow 1 \\
C_1 & \longrightarrow & \ldots & \longrightarrow & C_i & \longrightarrow & A \\
\uparrow & & & & \uparrow & & \uparrow \\
B & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

defines a map $A \to B[i]$ in the derived category $D(\mathcal{A})$. With the right notion of equivalence classes of exact sequences, this defines a bijective correspondence, due to N. Yoneda.

Ext

Example

Let B be a $\mathbb{Z}$-module and $n \geq 2$. There is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu_n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

yielding a commutative diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\mu_n} & B & \longrightarrow & B/nB & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}(\mathbb{Z}, B) & \xrightarrow{\mu_n^*} & \mathrm{Hom}(\mathbb{Z}, B) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) & \longrightarrow & 0 \end{array}$$

with exact rows. Hence, $B/nB \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B)$ is an isomorphism.

Stable module categories

Construction of a stable module category

1. Let G be a finite group, with group ring $k[G]$ over some field k.
2. Let $\mathrm{Mod}(k[G])$ be the category of left $k[G]$-modules.
3. Call two morphisms $f, g : A \to B$ in $\mathrm{Mod}(k[G])$ equivalent if $f - g$ factors through a projective module.
4. Consider equivalence classes of maps to obtain the stable module category $\mathrm{StMod}(k[G])$.
5. Then $\mathrm{StMod}(k[G])$ is a Frobenius category, that is, injectives coincide with projectives.
6. Consider the triangulated structure on $\mathrm{StMod}(k[G])$ with suspension ΣM of a module M given by $\Sigma M = \mathrm{coker}(M \to I)$, where I is the injective hull, and distinguished triangles coming from exact sequences.

Group cohomology and the stable module category

Definition

Let G be a group with group ring $\mathbb{Z}[G]$ and let A be a $\mathbb{Z}[G]$-module. Consider $\mathbb{Z}$ with the trivial G-action. The group cohomology $H^i(G, A)$ of G with coefficients in A is defined as

$$H^i(G, A) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Note that $H^0(G, A) = \mathrm{Hom}(\mathbb{Z}, A) = A^G$ is the submodule of G-invariant elements. In other words, the functor $A \mapsto A^G$ on $\mathbb{Z}[G]$-modules is left exact, so group cohomology is its right derived functor.

If G is finite, k is a field and A is a $k[G]$-module, then $H^i(G, A) \cong \mathrm{Ext}^i_{k[G]}(k, A)$. There is a cup product

$$H^i(G, k) \otimes H^j(G, k) \longrightarrow H^{i+j}(G, k),$$

yielding a graded commutative ring $H^*(G, k)$.

Group cohomology and the stable module category

Let G be a finite group and k a field of characteristic p.

Theorem (Quillen)

An element of $H^*(G,k)$ is nilpotent if and only if its restriction to every elementary abelian p-subgroup is nilpotent.

Theorem (Quillen)

The Krull dimension of $H^*(G,k)$ is equal to the p-rank of G, that is, the largest r such that $(\mathbb{Z}/p\mathbb{Z})^r \leq G$.

Let $V_G = \max H^*(G,k)$ be the spectrum of maximal ideals in $H^*(G,k)$. For A a finite-dimensional $k[G]$-module, $\mathrm{Ext}^*_{k[G]}(A,A)$ is a graded $H^*(G,k)$-module, so its annihilator is an ideal of $H^*(G,k)$, which defines a subvariety $V_G(A) \subset V_G$.

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