Lie Algebras – Lecture 3

M. Reineke

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Overview

Aim

- *Finish classification of semisimple Lie algebras: uniqueness, existence*
- *Adjoint group of a semisimple Lie algebra*
- *Universal enveloping algebra*
- $\mathbb{Z}$*-form*
- *Irreducible representations*

Summary of previous talk

We have extracted from a semisimple Lie algebra $\mathfrak{g}$ (with a choice of Cartan subalgebra $\mathfrak{h}$ = maximal abelian subalgebra of ad-diagonalizables) first its root system $(\mathbb{E}, \Phi)$ via Cartan decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \underbrace{\mathfrak{g}_\alpha}_{\text{dim}=1},$$

$\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Q}} (\langle \Phi \rangle_{\mathbb{Q}} \subset \mathfrak{h}^*)$, Φ finite, stable under reflections, integrality property,

then its Dynkin diagram determining the root system, and these admit a discrete classification.

It remains (!) to go backwards: show that the Dynkin diagram determines the Lie algebra up to isomorphism, and that any Dynkin diagram admits a corresponding semisimple Lie algebra.

Example of root system: $\mathfrak{sl}_n(k)$

$\mathfrak{sl}_n(k) = \{A \in \mathfrak{gl}_n(k) \,|\, \mathrm{tr}(A) = 0\}$ with basis

$$E_{ii} - E_{i+1,i+1} \text{ for } i = 1, \ldots, n-1 \text{ and } E_{ij} \text{ for } i \neq j.$$

$\mathfrak{h} = \mathrm{Ker}(\mathrm{tr}) \subset \mathfrak{t}_n$, thus $\mathfrak{h}^* \simeq \mathfrak{t}_n^* / \langle \mathrm{tr} \rangle$.

$[E_{kk}, E_{ij}] = (\delta_{ik} - \delta_{jk}) E_{ij}$, thus $E_{ij} \in (\mathfrak{sl}_n(k))_{E_{ii}^* - E_{jj}^*}$ for all $i \neq j$.

Thus $\mathbb{E} = \mathbb{R}^n / \langle \sum_i e_i \rangle$ and $\Phi = \{e_i - e_j \,|\, i \neq j\}$.

$\Delta = \{e_1 - e_2, \ldots e_{n-1} - e_n\}$ is a base, and

$$(e_i - e_{i+1}, e_j - e_{j+1}) = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}.$$

Yields Dynkin diagram A_{n-1}.

$\mathfrak{g}$ semisimple with Cartan $\mathfrak{g}$, associated roots $\Phi \subset \mathfrak{h}^*$, base $\Delta \subset \Phi$.

Theorem

Then $\mathfrak{g}$ is generated by the $\mathfrak{g}_\alpha$ for $\alpha \in \pm\Delta$. Consequently (!), the root systems determines the Lie algebra with its Cartan subalgebra.

Theorem

All Cartan subalgebras are conjugate under the adjoint group. Consequently (!), root system intrinsic to $\mathfrak{g}$.

Adjoint group: subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by all

$$\exp \mathrm{ad}_x = \sum_{n \geq 0} \frac{1}{n!} (\mathrm{ad}_x)^n$$

for ad_x nilpotent.

Universal enveloping algebra – definition

Functor $C : \mathrm{Assoc}_k \to \mathrm{Lie}_k$ maps $(A, \cdot)$ to $(A, [x, y] = x \cdot y - y \cdot x)$.

Admits left adjoint U: for any Lie algebra $\mathfrak{g}$ exists associative algebra $U\mathfrak{g}$, together with map $i : \mathfrak{g} \to U\mathfrak{g}$ such that $i([x, y]) = i(x) \cdot i(y) - i(y) \cdot i(x)$, with universal property:

for all $f : \mathfrak{g} \to A$ such that $f([x, y]) = f(x) \cdot f(y) - f(y) \cdot f(x)$, exists a unique map $\hat{f} : U\mathfrak{g} \to A$ such that $\hat{f} \circ i = f$:

$$\begin{array}{ccc} \mathfrak{g} & \stackrel{i}{\to} & U\mathfrak{g} \\ & f \searrow & \downarrow \hat{f} \\ & & A \end{array}$$

Universal enveloping algebra – construction

Construction: $T\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ tensor algebra (product is concatenation of tensors).

$$U\mathfrak{g} = T\mathfrak{g}/(x \otimes y - y \otimes x - [x, y]).$$

Theorem (PBW)

$i : \mathfrak{g} \to U\mathfrak{g}$ is injective. If $(x_i)_{i \in I}$ basis of $\mathfrak{g}$, then $\prod_{i \in I} x_i^{m_i}$ basis of $U\mathfrak{g}$.

Serre presentation

Theorem

$\mathfrak{g}$ semisimple, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra, $\Delta \subset \Phi \subset \mathfrak{g}^*$ (base of) roots, $\langle \_, \_ \rangle$ Cartan matrix on Δ. Then $U\mathfrak{g}$ is given by generators $h_\alpha, e_\alpha, f_\alpha$ for $\alpha \in \Delta$ subject to the relations:

- $[h_\alpha, h_\beta] = 0$,
- $[e_\alpha, f_\alpha] = h_\alpha$, $[e_\alpha, f_\beta] = 0$ for $\alpha \neq \beta$,
- $[h_\alpha, e_\beta] = \langle \beta, \alpha \rangle e_\beta$, $[h_\alpha, f_\beta] = -\langle \beta, \alpha \rangle f_\beta$,
- $(\mathrm{ad} e_\alpha)^{1-\langle \beta, \alpha \rangle} e_\beta = 0$,
- $(\mathrm{ad} f_\alpha)^{1-\langle \beta, \alpha \rangle} f_\beta = 0$.

Conversely, starting from Cartan matrix (or Dynkin diagram), this defines (the universal enveloping algebra of) a semisimple Lie algebra with this Dynkin diagram.

Examples: $\mathfrak{sl}_2$, $\mathfrak{sl}_3$

Already in first talk: $\mathfrak{sl}_2(k)$ has generators h, e, f and relations

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f.$$

From Serre presentations: $\mathfrak{sl}_3(k)$ has generators $h_1, h_2, e_1, e_2, f_1, f_2$. Namely, $h_1 = E_{11} - E_{22},\ h_2 = E_{22} - E_{33},\ e_1 = E_{12},\ e_2 = E_{23},\ f_1 = E_{21},\ f_2 = E_{32}$.

$\langle h_1, e_1, f_1 \rangle \simeq \langle h_2, e_2, f_2 \rangle \simeq \mathfrak{sl}_2(k)$.

Only interesting Serre relation: $[e_1, [e_1, e_2]] = 0$. Follows from $[e_1, e_2] = E_{13}$.

$\mathbb{Z}$-form

Theorem

$\mathfrak{g}$ semisimple: exist choice of $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$ such that

$$B = \{h_\alpha,\ \alpha \in \Delta,\ x_\alpha,\ \alpha \in \Phi\}$$

is basis with integral structure constants.

Thus for any field K we can define $\mathfrak{g}_K = K \otimes_{\mathbb{Z}} \langle B \rangle_{\mathbb{Z}}$. Not always semisimple!

Irreducible representations

From universal property of $U\mathfrak{g}$:

$$\mathrm{Rep}(\mathfrak{g}) \simeq \mathrm{Mod}\, U\mathfrak{g}.$$

$\mathfrak{g}$ solvable: all finite dimensional irreducible representation $\mathfrak{g} \to \mathfrak{gl}(V)$ (that is, only $\mathfrak{g}$-invariant subspaces of V are 0 and V) are one-dimensional, parametrized by $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

But extend nontrivially; in general $\mathrm{Ext}^1_{\mathfrak{g}}(V, W) \neq 0$.

Theorem (Weyl)

All finite-dimensional representations of $\mathfrak{g}$ semisimple are completely reducible, that is, direct sums of irreducibles.

Irreducible representations

Theorem

The irreducible $\mathfrak{g}$*-representations are parametrized (up to iso) by the dominant integral weight* $\lambda \in \mathfrak{h}^*$ *such that* $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ *for all* $\alpha \in \Delta$*.*

Construction:

$$V(\lambda) = U\mathfrak{g}/U\mathfrak{g} \cdot \langle e_\alpha,\ h_\alpha - \lambda(h_\alpha)1,\ f_\alpha^{\langle \lambda, \alpha \rangle + 1}\ \alpha \in \Delta \rangle.$$

Final slide

Thank you!