Lie Algebras – Lecture 2

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Overview

Aim

- *Formulate Cartan-Killing classification of (semi-)simple Lie algebras*
- *Extract root system from a semisimple Lie algebra*
- *Extract Dynkin diagram from root system*

Summary of previous talk

- Definition of Lie algebras, discussion of the axioms,
- Lie algebras from associative algebras, $\mathfrak{gl}_n(k)$,
- Lie algebras of derivations, $\mathrm{Der}_k(k[X])$, $\mathfrak{g}_2 = \mathrm{Der}_k(\mathbb{O})$,
- Lie algebras of algebraic groups $\mathrm{Lie}(G) = \mathrm{Der}_k(k[G])^G \simeq T_1(G)$,
- $\mathfrak{sl}_n(k)$, $\mathfrak{so}_n(k)$, $\mathfrak{sp}_{2n}(k)$; $\mathfrak{t}_n \oplus \mathfrak{n}_n = \mathfrak{b}_n \subset \mathfrak{gl}_n$,
- lower central and derived series; abelian, nilpotent and solvable Lie algebras,
- radical, (semi-)simple Lie algebras, criteria for semisimplicity.

Levi decomposition

We have seen the canonical decomposition of a Lie algebra:

$$0 \to \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{solvable}} \to \mathfrak{g} \to \underbrace{\mathfrak{g}/\mathrm{rad}(\mathfrak{g})}_{\text{semisimple}} \to 0.$$

Even better:

Theorem (Levi)

Exists semisimple Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ *such that* $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{s}$ *as k-vector space. Thus* $\mathfrak{s} \simeq \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$, *and even* $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$.

Discussion of solvable Lie algebras

Why will we ignore the solvable part and concentrate on semisimple Lie algebras?

- We know where to find (all) solvable Lie algebras: they all sit as Lie subalgebras in some $\mathfrak{b}_n$, and, conversely, any Lie subalgebra of $\mathfrak{b}_n$ is solvable.
- Even better, any solvable Lie algebra is filtered by trivial Lie algebras: exists

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^n = 0$$

  sequence of ideals, all subquotients $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ one-dimensional.

But a classification up to isomorphism is "'hopeless".

Criteria for semisimplicity

Theorem

Equivalent:

- $\mathfrak{g}$ *semisimple*
- *Killing form* $\kappa(x, y) = \mathrm{tr}\left([x, \_] \circ [y, \_] \in \mathrm{End}(\mathfrak{g})\right)$ *nondegenerate*
- $\mathfrak{g} \simeq \bigoplus_{i=1}^{n} \mathfrak{g}_i$, *all* $\mathfrak{g}_i$ *simple.*

Cartan-Killing classification

Theorem

A Lie algebra is simple if and only if it is isomorphic to one of following:

- *one in the infinite series* $\mathfrak{sl}_n(k)$, $\mathfrak{so}_n(k)$, $\mathfrak{sp}_{2n}(k)$,
- *one of five exceptional Lie algebras* $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

Root space decomposition - adjoint representation

From now on, $\mathfrak{g}$ semisimple.

Idea: Analyze $\mathfrak{g}$ via its *adjoint representation*.

Definition

Representation of a Lie algebra $\mathfrak{g}$ on k-vector space V is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$.

$x \in \mathfrak{g}$: $\mathrm{ad}_x = [x, \_] \in \mathrm{End}(\mathfrak{g})$.

Defines *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ \mathrm{ad}(x) = \mathrm{ad}_x$.

Recall from linear algebra: *commuting* diagonalizable operators $\varphi_1, \ldots, \varphi_k \in \mathrm{End}_k(V)$ are simultaneously diagonalizable:

$$V = \bigoplus_{\lambda_1,\ldots,\lambda_k} V_{\lambda_1,\ldots,\lambda_k},$$

$$V_{\lambda_1,\ldots,\lambda_k} = \{v \in V \mid \varphi_i(v) = \lambda_i v,\ i = 1, \ldots, k\}.$$

Root space decomposition - Cartan decomposition

Choose (!!) maximal family $x_1, \ldots, x_k \in \mathfrak{g}$ of commuting elements such that ad_{x_i} diagonalizable. $\mathfrak{h} = \langle x_1, \ldots, x_k \rangle$ *Cartan subalgebra.* Consider simultaneous eigenspace decomposition:

Definition (Cartan decomposition)

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x,\ h \in \mathfrak{h}\}.$$

Facts

- $\mathfrak{g}_0 = \mathfrak{h}$.
- $\Phi := \{0 \neq \alpha \in \mathfrak{h}^* \,|\, \mathfrak{g}_\alpha \neq 0\} \subset \mathfrak{h}^*$ *finite.*
- $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.

Root space decomposition - extracting the root system

Facts

- *κ nondegenerate on $\mathfrak{h}$, thus induces $(\_,\_)$ on $\mathfrak{h}^*$,*
- *Φ spans $\mathfrak{h}^*$. Define $\mathbb{E}_{\mathbb{Q}} = \langle \Phi \rangle_{\mathbb{Q}} \subset \mathfrak{h}^*$ (since $\mathbb{Q} \subset k$).*
- *$(\_,\_)$ positive definite on $\mathbb{E}_{\mathbb{Q}}$.*

Define $\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}$: *Euklidean* vector space with distinguished subset Φ.

The pair $(\mathbb{E}, \Phi)$ has very special rigidity properties, encoded in the axioms of a *root system*.

Root space decomposition - axioms of a root system

Theorem (Root system of a semisimple Lie algebra)

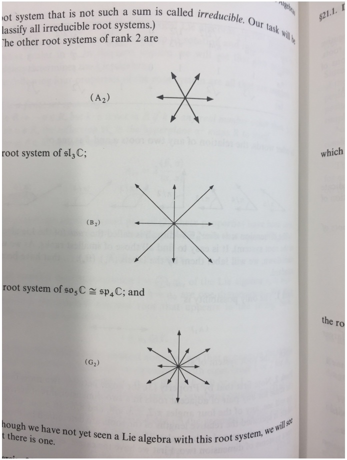
$(\mathbb{E}, \Phi)$ *is a root system, that is:*

- Φ *is finite, spans* $\mathbb{E}$*, does not contain* 0.
- *If* $\alpha \in \Phi$*, then* $k\alpha \in \Phi$ *iff* $k = \pm 1$.
- Φ *stable under reflections at hyperplanes orthogonal to* Φ*:*

$$\alpha \in \Phi \text{ implies } \sigma_\alpha(\Phi) \subset \Phi, \text{ where } \sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

- *Integrality: If* $\alpha, \beta \in \Phi$*, then* $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Examples of roots systems



ot system that is not such a sum is called *irreducible*. Our task will be
lassify all irreducible root systems.)
he other root systems of rank 2 are

(A_2)

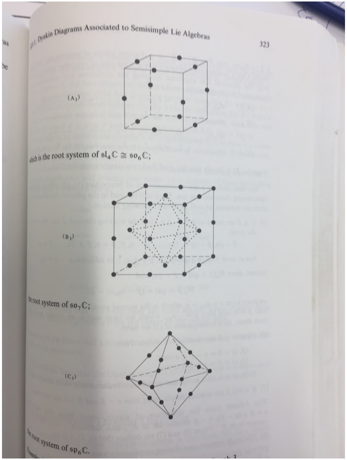
root system of $\mathfrak{sl}_3\mathbb{C}$;

(B_2)

root system of $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$; and

(G_2)

hough we have not yet seen a Lie algebra with this root system, we will see
t there is one.



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(A_3)

which is the root system of $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$;

(B_3)

the root system of $\mathfrak{so}_7\mathbb{C}$;

(C_3)

the root system of $\mathfrak{sp}_6\mathbb{C}$.

Classification of root systems

Facts

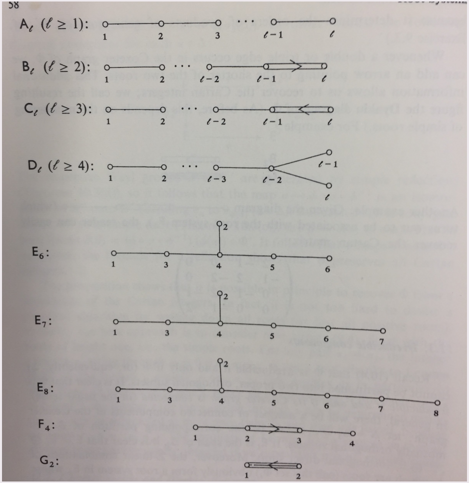
- *Φ admits a base, that is: exists $\Delta \subset \Phi$, basis of $\mathbb{E}$, $\Phi \subset \mathbb{N}\Delta \cup -\mathbb{N}\Delta$,*
- *the Cartan matrix $C = (\langle \beta, \alpha \rangle)_{\alpha,\beta \in \Delta}$ determines $(\mathbb{E}, \Phi)$ completely.*
- *$C \in M_n(\mathbb{Z})$ fulfills:*
    - *$C_{ii} = 2$,*
    - *$C_{ij} \leq 0$ for $i \neq j$,*
    - *C symmetrizable: $C = DS$, D diagonal, S symmetric,*
    - *C positive definite.*

Encode C completely in *Dynkin diagram*: graph with vertices Δ, number of edges between α and β is $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle$, arrow from α to β if $\langle \alpha, \beta \rangle < \langle \beta, \alpha \rangle$.

Classification of root systems

Theorem

Any Dynkin diagram is a disjoint union of the following:



Completion of the classification

We have extracted from a semisimple Lie algebra $\mathfrak{g}$ (with a choice of Cartan subalgebra $\mathfrak{h}$) first its root system $(\mathbb{E}, \Phi)$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \underbrace{\mathfrak{g}_\alpha}_{\dim = 1},$$

then its Dynkin diagram determining the root system, and these admit a discrete classification.

It remains (!) to go backwards: show that the Dynkin diagram determines the Lie algebra up to isomorphism, and that any Dynkin diagram admits a corresponding semisimple Lie algebra.

Final slide

Thank you!