Lie Algebras – Lecture 1

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Overview

Aim

- *Introduce main classes of Lie algebras*
- *Discuss lots of examples*
- *Structure theory from general to (semi)simple Lie algebras*

k always denotes algebraically closed field of characteristic 0.

Definitions

Definition

Lie algebra: k-vector space $\mathfrak{g}$ with bilinear *bracket* $[\_,\_] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[y,x] = -[x,y]$ for all $x,y \in \mathfrak{g}$ (*antisymmetry*)
- $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ for all $x,y,z \in \mathfrak{g}$ (*Jacobi identity*)

Discussion of definitions

$$[y, x] = -[x, y],$$
$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

Remark

In other words, $[\_, \_] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$.

How to memorize the Jacobi identity?

Definition: *Derivation of an (nonassoc.) algebra* A *: linear map* $d : A \to A$ *such that* $d(ab) = d(a)b + ad(b)$ *for* $a, b \in A$ *(Leibniz rule).*

Jacobi identity means: any $[x, \_]$ *is a derivation on* $\mathfrak{g}$.

Alternatively: $[x, [y, z]] +$ *cyclic shifts* $= 0$.

Some standard definitions

Definition

- Lie subalgebra: subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
- Homomorphism of Lie algebras: linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.
- $I \subset \mathfrak{g}$ ideal if $[\mathfrak{g}, I] \subset I$.

As usual, then $\mathfrak{g}/I$ is Lie algebra.

And $\mathfrak{g}/\mathrm{ker}(\varphi) \simeq \mathrm{im}(\varphi)$ as Lie algebras

Examples 1: Lie algebras from associative algebras

Examples

A associative k-algebra: A is Lie algebra via $[x, y] = xy - yx$.

In particular $A = \mathrm{End}_k(V)$: as Lie algebra called $\mathfrak{gl}(V)$.

In particular $A = M_n(k)$: as Lie algebra called $\mathfrak{gl}_n(k)$.

Linear basis of $\mathfrak{gl}_n(k)$: E_{ij} for $i, j = 1, \ldots, n$, with bracket

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}.$$

A Lie subalgebra of some $\mathfrak{gl}_n(k)$ is called a *linear* Lie algebra.

Examples 2: Lie algebras of derivations

Examples

A arbitrary (even nonassoc.) algebra:

$\mathrm{Der}_k(A) = \{d : A \to A \text{ derivation}\} \subset \mathfrak{gl}(A)$ Lie subalgebra!

In particular X (smooth) affine variety: $\mathrm{Der}_k(k[X])$ *Lie algebra of vector fields on* X.

Concrete example: $\mathrm{Der}_k(k[t]) = k[t]\frac{d}{dt}$ (infinite-dimensional!)

with bracket $[P\frac{d}{dt}, Q\frac{d}{dt}] = (PQ' - P'Q)\frac{d}{dt}$.

Basis $e_n = t^n\frac{d}{dt}$: bracket $[e_m, e_n] = (n-m)e_{m+n-1}$.

Examples 3: an exotic Lie algebra of derivations

Examples

Nonassociative algebra of Cayley octonions: $\mathbb{O} = \begin{bmatrix} k & k^3 \\ k^3 & k \end{bmatrix}$ with multiplication

$$\begin{bmatrix} \lambda & v \\ w & \mu \end{bmatrix} * \begin{bmatrix} \lambda' & v' \\ w' & \mu' \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda\lambda' - v \cdot w' & \lambda v' + \mu' v + w \times w' \\ \lambda' w + \mu w' + v \times v' & \mu\mu' - w \cdot v' \end{bmatrix}.$$

$\mathrm{Der}_k(\mathbb{O}) =: \mathfrak{g}_2$ an *exceptional simple* Lie algebra.

Examples 4: Lie algebra of algebraic group, abstract

Examples

G affine algebraic group: $k[G]$ carries G-action via $(gf)(h) = f(hg)$. Then $\mathrm{Der}_k(k[G])$ contains

$$\mathrm{Lie}(G) = \{d \in \mathrm{Der}_k(k[G]) : d(gf) = gd(f) \text{ for all } g \in G\}$$

Lie algebra of invariant vector fields.

$1 \in G$ corresponds to maximal ideal $\mathfrak{m} \subset k[G]$:

$$\mathrm{Lie}(G) \simeq (\mathfrak{m}/\mathfrak{m}^2)^* = T_1(G),$$

Lie algebra structure on tangent space to 1 in G.

Examples 5: Lie algebra of algebraic group, concrete

Examples

$G \subset \mathrm{GL}_n(k)$ closed subgroup: linear algebraic group.

$$T_1(G) \subset T_1(\mathrm{GL}_n(k)) \simeq \mathfrak{gl}_n(k) \text{ (as Lie algebra)}$$

is a Lie subalgebra!

$$T_1(G) = \{x \in \mathfrak{gl}_n(k) \,:\, f(1+\epsilon x) = 0 \text{ for all } f \in \mathbb{V}(G)\}$$

where $\epsilon^2 = 0$.

Examples

- $\mathrm{SL}_n(k) = \{g \in \mathrm{GL}_n(k) \ : \ \det(g) = 1\} \subset \mathrm{GL}_n(k)$.

$$\det(1 + \epsilon x) = 1 + \epsilon tr(x) \bmod \epsilon^2$$

  $\mathrm{Lie}(\mathrm{SL}_n(k)) = \{x \in \mathfrak{gl}_n(k) \ : \ \mathrm{tr}(x) = 0\} = \mathfrak{sl}_n(k)$.
- $\mathrm{SO}_n(k) = \{g \in \mathrm{GL}_n(k) \ : \ gg^t = 1, \det(g) = 1\} \subset \mathrm{GL}_n(k)$.

$$1 = (1 + \epsilon x)(1 + \epsilon x^t) = 1 + \epsilon(x + x^T).$$

  $\mathrm{Lie}(\mathrm{SO}_n(k)) = \{x^T = -x\} =: \mathfrak{so}_n(k) \subset \mathrm{gl}_n(k)$.
- Similarly

  $\mathfrak{sp}_{2n}(k) = \{x^T J = -Jx\} \subset \mathfrak{gl}_{2n}(k), \ J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}$.

Examples 7: even more linear Lie algebras

Examples

- $\mathfrak{b}_n$ upper triangular matrices,
- $\mathfrak{n}_n$ strictly upper triangular matrices,
- $\mathfrak{t}_n$ diagonal matrices.

$$\mathfrak{t}_n \oplus \mathfrak{n}_n = \mathfrak{b}_n \subset \mathfrak{gl}_n.$$

Examples 8: two prominent three-dimensional Lie algebras

Examples

Heisenberg Lie algebra: basis x, y, z, bracket $[x, y] = z$, $[z, \_] = 0$.

Isomorphic to $\mathfrak{n}_3$.

$\mathfrak{sl}_2(k)$ has basis $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

bracket $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Realized by derivations on $k[X, Y]_n$ (homogeneous polynomials of degree n) via

$$E = Y\frac{\partial}{\partial X},\ F = X\frac{\partial}{\partial Y},\ H = Y\frac{\partial}{\partial Y} - X\frac{\partial}{\partial X}.$$

(Irreducible representations of $\mathfrak{sl}_2(k)$).

Definition

Commutator subalgebra: $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$.

Lower central series: $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$ for $n \geq 1$.

Derived series: $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}\mathfrak{g}^{(n-1)}]$ for $n \geq 1$.

Theorem

$\mathfrak{g}$ *finite dimensional:*

$\mathfrak{g}$ *abelian, i.e.* $[\mathfrak{g}, \mathfrak{g}] = 0$, *iff* $\mathfrak{g} \hookrightarrow \mathfrak{t}_n$ *for some* n.

Engel $\mathfrak{g}$ *nilpotent, i.e. lower central series terminates, iff* $\mathfrak{g} \hookrightarrow \mathfrak{n}_n$ *for some* n.

Lie $\mathfrak{g}$ *solvable, i.e. derived series terminates, iff* $\mathfrak{g} \hookrightarrow \mathfrak{b}_n$ *for some* n.

Ado *Any* $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ *for some* n.

Radical and semisimplicity

Definition

$\mathfrak{g}$ simple if it has no ideals except $0, \mathfrak{g}$.

Any $\mathfrak{g}$ has unique maximal solvable ideal $\mathrm{rad}(\mathfrak{g}) \subset \mathfrak{g}$, the radical of $\mathfrak{g}$.

$\mathfrak{g}$ semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$.

Canonical decomposition into solvable and semisimple part:

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to 0.$$

Theorem

Equivalent:

- $\mathfrak{g}$ *semisimple*
- *Killing form* $(x, y) \mapsto \mathrm{tr}([x, \_] \circ [y, \_] \in \mathrm{End}(\mathfrak{g}))$ *nondegenerate*
- $\mathfrak{g} \simeq \bigoplus_{i=1}^n \mathfrak{g}_i$, *all* $\mathfrak{g}_i$ *simple.*

Final slide

Thank you!