PART 1: The inverse Galois problem over $\mathbb{C}(t)$

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The goal of the first half of the seminar is to prove the following theorem.

Theorem. *For every finite group G, there is a Galois field extension $L|\mathbb{C}(t)$ such that* $\mathrm{Gal}(L|\mathbb{C}(t)) \cong G$.

We will mainly follow the book [SZ]. If you have suggestions for other related topics, please let me know and I can try to include them in the programme. Finally, if you have any questions about the programme or the preparation of your talk please let me know.

TALK 1: Basics and preliminaries. *Chapter 1 of [SZ].*

This talk recalls the basics of Galois groups and Etale algebras necessary for the rest of the seminar.

Start with Proposition 1.3.5, briefly mention the Krull topology on a Galois group. State Theorem 1.3.11. State Theorem 1.5.4. Feel free to skip some proofs, but concentrate on giving a general idea of this correspondence.[1]

TALK 2: Galois covers of topological spaces. *Chapter 2 of [SZ].*

This talk deals with the basics on manifold and covering theory. The main idea is to introduce "Galois covers" of topological spaces and see that there is a "fundamental theorem of Galois covers", parallel to the one on Galois field extensions.

Cover Sections 2.1 and 2.2 in detail. Finish with Example 2.4.12 and the definition of fiber product of spaces (just after Example 2.4.12).

TALK 3: Riemann surfaces and examples. *Chapter 3 of [SZ].*

This talk surveys basic notions and properties of Riemann surfaces. Here we will see the notion of "branched Galois cover", i.e. a topological Galois cover with some bad points.

Go through Sections 3.1 and 3.2 in full.

TALK 4: Coverings of Riemann surfaces and field theory. *Chapter 3 of [SZ].*

This talk connects the worlds of Galois theory and Riemann surfaces. It turns out that the ring of meromorphic functions of a connected Riemann surface is a field, and Galois covers yield Galois field extensions.

Cover Section 3.3.

TALK 5: The absolute Galois group of $\mathbb{C}(t)$. *Chapter 3 of [SZ].*

In this last talk of the first part we will see the solution of the inverse Galois problem over $\mathbb{C}(t)$.

Cover Section 3.4. If necessary skip parts of the proof of Theorem 3.4.8.

[1] If needed, we could spend some time in the PhD-Postdoc meetings on profinite groups, the Krull topology and the many examples appearing in Chapter 1.

PART 2: Etale Fundamental Groups

Programme prepared by: Thuong Dang

TALK 6: Galois Theory for Finite Etale Covers. *Chapter 5 of [SZ].*

Recall some basic facts about schemes and etale morphisms of schemes. State Proposition 5.3.8 of [SZ] about finite Galois covers, and its analogue in the topological/Galois theory context.

TALK 7: Etale Fundamental Groups of Schemes. *Chapter 5 of [SZ].*

Define fiber functors and state Theorem 5.4.2 of [SZ] about their pro-representability. Discuss the example of Spec(k), then go to Proposition 5.4.9 about the fundamental group of an integral normal scheme with generic base point. See also Section 53.6, Section 53.11 of [ST].

TALK 8: Examples of Etale Fundamental Groups. *Chapter 4 of [SZ].*

Give examples of etale fundamental groups of schemes, e.g. Spec($\mathbb{Z}$) and normal curves, as discussed in Chapter 4 of [SZ]. State Theorem 4.6.7, Theorem 4.6.10 and Example 4.6.12 in [SZ]. Feel free to add more examples.

TALK 9: An application: the Inverse Galois Problem over $\mathbb{Q}$. *Chapter 4 of [SZ].*

Go through Section 4.8. Try to give a general idea of the methods used. It would be nice to see at least Example 4.8.10 in detail, where $\mathrm{PSL}_2(\mathbb{F}_p)$ is realised as a Galois group over $\mathbb{Q}$.

TALK 10: The Homotopy Exact Sequence and Applications. *Chapter 5 of [SZ].*

This talk will cover almost all of Section 5.6 of [SZ]. First, state Proposition 5.6.4 about the homotopy exact sequence. Discuss Corollary 5.6.6 and its applications (Proposition 5.6.7, Proposition 5.6.8, and Corollary 5.6.9 about abelian varieties). See also Section 53.15 of [ST].

References

[ST] Stack Project, https://stacks.math.columbia.edu/, Johan de Jong et al.

[SZ] T. Szamuely. *Galois groups and fundamental groups.* Cambridge studies in advanced Mathematics 117, 2009.