GRK Ring Lecture: Deformation Theory, Part II

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Set up:

Throughout:

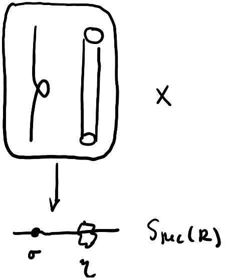
- Let k be a field,
- X_0 a k-scheme of finite type,
- R a local noetherian ring with residue field $R/\mathfrak{m}_R = k$.

We seek to understand **deformations** of X_0 over the ring R.

This are pairs (X, φ) where X is a **flat** R-scheme of finite type, and $\varphi : X_0 \to X \otimes_R k$ is an isomorphism.

Too difficult

If R is a discrete valuation ring, like $R = k[[t]]$, it looks so:



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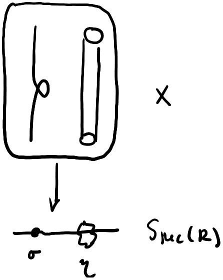
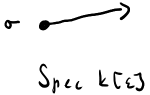


That is **far too difficult for us**, at least at the moment! What is a simpler choice for the ring R?

The ring of dual numbers

We work over the **ring of dual numbers** $R = k[\epsilon]$, where ϵ is a formal symbol subject to $\epsilon^2 = 0$.

This is a **local Artin ring** with residue field k. It also has a k-algebra structure. The spectrum is a singleton $\{\sigma\}$, with a tangent vector attached.



Deformations of X_0 over the ring $R = k[\epsilon]$ are called **first-order deformations**.

Deformations over dual numbers

Let (X, φ) be a deformation over the dual numbers $R = k[\epsilon]$. It sits in a cartesian square

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

Hence the inclusion $X_0 \subset X$ is a **homeomorphism**, so there is **no topology** left in our problem!

Deformations over dual numbers

Flatness ensures that in the exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X_0} \longrightarrow 0,$$

the ideal is $\mathscr{I} = k\epsilon \otimes_k \mathscr{O}_{X_0} = \epsilon\mathscr{O}_{X_0}$.

In particular $\mathscr{O}_X$ is an **extension** of $\mathscr{O}_{X_0}$ by $\mathscr{I} = \epsilon\mathscr{O}_{X_0}$, viewed as coherent sheaves on X.

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We would prefer to have extensions with **coherent sheaves on** X_0!

Kähler differentials

Recall that each A-algebra B comes with a B-module of **Kähler differentials** $\Omega^1_{B/A}$, defined by the short exact sequence

$$0 \longrightarrow \Omega^1_{B/A} \longrightarrow (B \otimes_A B)/I^2 \longrightarrow (B \otimes B)/I \longrightarrow 0$$

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Similar definition applies for morphisms of schemes. In particular we get a quasicoherent sheaf $\Omega^1_{X_0/k}$. The dual

$$\Theta_{X_0/k} = \underline{\mathrm{Hom}}(\Omega^1_{X_0/k}, \mathscr{O}_{X_0})$$

is the **tangent sheaf**.

The standard exact sequence

The k-structure on dual numbers gives $X_0 \to X \to \mathrm{Spec}(k)$, yields **standard exact sequence**

$$\ldots \to \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{X/k} \otimes \mathscr{O}_{X_0} \to \Omega^1_{X_0/k} \to \Omega^1_{X_0/X} \to 0,$$

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with for Kähler differentials. The map on the left is $[f] \mapsto df \otimes 1$.

Fact: If the scheme X_0 is reduced and generically smooth, the map $\mathscr{I}/\mathscr{I}^2 \to \Omega^1_{X/k}$ is injective. In any case, $\Omega^1_{X_0/X} = 0$.

Kodaira–Spencer

Suppose from now that X_0 is reduced and generically smooth. The deformation (X, φ) gives an **extension**

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{X/k} \otimes \mathscr{O}_{X_0} \to \Omega^1_{X_0/k} \to 0$$

of coherent sheaves on X_0. Yields **Yoneda class**

$$[\Omega^1_{X/k} \otimes \mathscr{O}_{X_0}] \in \mathrm{Ext}^1(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0}).$$

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Theorem. *The mapping* $(X, \varphi) \mapsto [\Omega^1_{X/k} \otimes \mathscr{O}_{X_0}]$ *identifies isomorphism classes of deformations of* X_0 *over the dual numbers* $R = k[\epsilon]$ *with vectors in* $\mathrm{Ext}^1(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0})$.

Idea of proof

We describe the inverse mapping:

Suppose we have an extension $\mathscr{E}$. The universal differential $f \mapsto df$ defines via cartesian square

$$\begin{array}{ccccccccc} & & & & \mathscr{O}_X & \longrightarrow & \mathscr{O}_{X_0} & & \\ & & & & \downarrow & & \downarrow d & & \\ 0 & \longrightarrow & \epsilon\mathscr{O}_{X_0} & \longrightarrow & \mathscr{E} & \longrightarrow & \Omega^1_{X_0/k} & \longrightarrow & 0 \end{array}$$

an abelian sheaf $\mathscr{O}_X$. One specifies multiplication as in dual numbers, using $d(fg) = fdg + gdf$.

The ringed space $X = (X_0, \mathscr{O}_X)$ becomes the total space of the deformation.

Applications

If X_0 is smooth, $\Omega^1_{X_0/k}$ is locally free, with dual $\Theta_{X_0/k}$, and we get an identification

$$\mathrm{Ext}^1(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0}) = H^1(X_0, \Theta_{X_0/k}).$$

with cohomology of the tangent sheaf.

Cohomology groups are **more amenable to computations** that Ext groups. In any case, the zero class corresponds to the **constant deformation**

$$X = X_0 \otimes_k k[\epsilon]$$

Applications

Corollary. *If the scheme X_0 is smooth and affine, then every deformation over $R = k[\epsilon]$ is isomorphic to the constant deformation.*

Proof: Use Serre's Cohomological Criterion for affineness.

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Corollary. *Ever deformation of the projective space $X_0 = \mathbb{P}^n$ over $R = k[\epsilon]$ is isomorphic to the constant deformation.*

Proof: Use the Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \oplus_{i=0}^{n} \mathscr{O}_{\mathbb{P}^n}(1) \longrightarrow \Theta_{\mathbb{P}^n/k} \longrightarrow 0.$$

Applications

Corollary. *Let $X = C$ be a smooth curve of genus $g \geq 2$. Then the space of first order deformations has dimension $d = 3g - 3$.*

Proof: The structure sheaf has $\chi(\mathscr{O}_C) = 1 - g$. The **dualizing sheaf** $\omega_C = \Omega^1_{C/k}$ has degree $r = 2g - 2$. Its inverse $\mathscr{L} = \Theta_{C/k}$ has degree $-r = 2 - 2g < 0$. **Riemann–Roch** gives

$$-h^1(\mathscr{L}) = \chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_C) = 3 - 3g.$$

Historical starting point:

This is the **starting point of algebraic geometry**! From Riemann's 1857 paper on abelian functions:

"... the corresponding class [...] depends on $3g - 3$ continuous variables, which we shall call the **moduli** of the class."

Bernhard Riemann (1826–1866)

Automorphisms and obstructions

The **automorphisms group** of the extension

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{X/k} \otimes \mathscr{O}_{X_0} \to \Omega^1_{X_0/k} \to 0$$

is the Hom group

$$\mathrm{Hom}(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0}) = \mathrm{Ext}^0(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0}).$$

By our Theorem, this is also the automorphism group for every first-order deformation (X, φ).

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We shall see later that the Ext group $\mathrm{Ext}^2(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0})$ containes the **obstructions** against higher-order deformations over rings like $R = k[t]/(t^n)$.

Formal schemes

Suppose $X_0 = C$ is a smooth curve. Then the obstruction group

$$\mathrm{Ext}^2(\Omega^1_{X_0/k}, \epsilon\mathscr{O}_{X_0}) = H^2(X_0, \Theta_{X_0/k})$$

is zero, by Grothendieck's Vanishing result.

So one finds compatible deformations (X_n, φ_n) of $X_0 = C$ over the rings $R_n = k[t]/(t^{n+1})$. This gives an inverse system of proper schemes $\mathfrak{X} = (X_n)_n \geq 0$ over the complete local rings $R = k[[t]]$. This are the so-called **formal schemes**.

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Does the formal R-scheme $\mathfrak{X}$ come from an R-scheme X, such that $X_n = X \otimes R_n$?

Grothendieck's Algebraization Theorem

Theorem. *Suppose there is a compatible family* $(\mathscr{L}_n)$ *of invertible sheaves on* (X_n)*, such that* $\mathscr{L}_0 \in \mathrm{Pic}(X_0)$ *is ample. Then there is a proper* R*-scheme* X *inducing the formal scheme* $\mathfrak{X}$*, unique up to unique isomorphism.*

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This holds for arbitrary formal schemes. In dimension one, each proper scheme is projective (Riemann–Roch). The obstruction to lift a class in $\mathrm{Pic}(X_n) = H^1(X_n, \mathscr{O}_{X_n}^\times)$ to the thickening X_{n+1} lies in $H^2(C, \mathscr{O}_C) \otimes kt^{n+1}$, which is zero!

So the $d = 3g - g$ first-oder deformations for C indeed give Riemann's moduli!

The case of smooth affine schemes

We now examine deformations over general local Artin rings R with residue field k and having a k-algebra structure.

Lemma. *If the scheme X_0 is smooth and affine, then every deformation over R is constant.*

Proof: Saw this already for $R = k[\epsilon]$. In general, apply induction on length(R), and use definition of smoothness.

The case of smooth affine schemes

Suppose that we have a deformation (X_{n-1}, φ_{n-1}) of X_0 over the ring $R_{n-1} = k[t]/(t^n)$.

Choose affine open covering $X_0 = U_1 \cup \ldots \cup U_r$. On each U_i the deformation over R_{n-1} becomes constant. So they extend to R_n. On overlaps U_{ij} these differ by isomorphisms φ_{ij}.

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May not satisfy **cocycle condition**, but $\varphi_{jk} \circ \varphi_{ij} = f_{ijk}\varphi_{ik}$ defines cocycle $f_{ijk} \in \Gamma(U_{ijk}, \Theta_{X_0/k} \otimes kt^n)$. Yields some cohomology class ob $\in H^2(X_0, \Theta_{X_0/k})$. Is the **obstruction** for changing the local isomorphisms so that global deformation arises via glueing.

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If ob $= 0$, the set of all deformations (X_n, φ_n) restricting to (X_{n-1}, φ_{n-1}) is a **torsor** under $H^1(X_0, \Theta_{X_0/k})$.

Formal smoothness for functors of Artin rings

Let Art(k) be the category of local Artin rings R with residue field k. Consider the functor

$$h : \mathsf{Art}(k) \longrightarrow (\mathrm{Set})$$

that sends R to the set of isomorphism classes of deformations (X, φ) of the scheme X_0 over the ring R.

Theorem. *Suppose that X_0 is smooth and $h^2(\Theta_{X_0/k}) = 0$. Then the above functor is formally smooth.*

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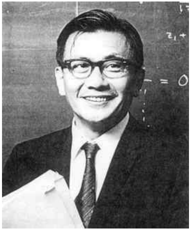
Here **formal smoothness of functors** means $h(A) \to h(A/I)$ is surjective for square-zero ideals, as in my first lecture. The result applies if $X_0 = C$ is a proper smooth curve.

K3 surfaces

It also applies if $X_0 = S$ is a **K3 surface** ($c_1 = 0$ and $b_2 = 22$).

Ernst Kummer
(1810–1893)



Kunihike Kodaira
(1915–1997)

Erich Kähler
(1906–2000)

Examples are quartic hypersurfaces $S \subset \mathbb{P}^3$, or the resolution of singularities for $S \to A/\{\pm 1\}$.

K3 surfaces

Although deformations of the scheme X_0 are unobstructed, there are obstructions for deforming invertible sheaves $\mathscr{L}_0$. They lie in $H^2(X_0, \mathscr{O}_{X_0}) = k$.

Over $k = \mathbb{C}$, this leads to **non-algebraic K3 surfaces**. Then the field of meromorphic functions $f : S \dashrightarrow \mathbb{C}$ has transcendence degree trdeg < 2.

For general ground fields k, this yields formal families $\mathfrak{X} \to \mathrm{Spec}(k[[t]])$ of K3 surfaces that are **not algebraizable**.

Thank you very much for the attention!