GRK Ring Lecture: Brauer groups and obstructions, Part II

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Picard and Brauer groups

Let X be a scheme. Recapitulation:

The **Picard group** consists of isomorphism classes of invertible sheaves $\mathscr{L}$. Can be seen as twisted forms of $\mathscr{O}_X$. Gives

$$\mathrm{Pic}(X) = H^1(X, \mathbb{G}_m).$$

The **Brauer group** comprises equivalence classes of Azumaya algebras $\mathscr{A}$. These are twisted forms for $\mathrm{Mat}_n(\mathscr{O}_X)$. Gives

$$\mathrm{Br}(X) \subset H^2(X, \mathbb{G}_m).$$

Cohomological interpretation

Recall **Grothendieck's cohomological interpretation** for Azumaya algebras $\mathscr{A}$:

Choose isomorphism $\varphi : \mathscr{A}|U \to \mathrm{Mat}_n(\mathscr{O}_X)|U$ on some flat surjective $U \to X$.

Write $(\varphi \otimes 1) = \psi \circ (1 \otimes \varphi)$ for some $\psi \in \Gamma(U^2, \mathrm{PGL}_n)$.

Choose lift $\tilde{\psi} \in \Gamma(U^2, \mathrm{GL}_n)$, after refining $U \to X$.

Gives 2-cocycle $\alpha = \tilde{\psi}_{12} \cdot \tilde{\psi}_{02}^{-1} \cdot \tilde{\psi}_{01} \in \Gamma(U^3, \mathbb{G}_m)$. Via Čech cohomology get desired class $[\alpha] = [\mathscr{A}] \in H^2(X, \mathbb{G}_m)$.

Projective schemes

Let k be a ground field, of characteristic $p \geq 0$. In algebraic geometry, it is **very easy to write down objects**:

Any system of homogeneous polynomial equations

$$f_i(T_0, \dots, T_n) = 0, \quad 1 \leq i \leq m$$

defines a closed subscheme $X \subset \mathbb{P}^n$.

If such a description is possible, one says X is projective.

Any projective scheme is proper. By Chow's Lemma, any proper scheme can be modified to a projective scheme.

Morphisms?

But it is **very difficult to specify morphisms** with a fixed domain X. Basically, there are only two methods, involving either groups or invertible sheaves:

First method: Given a finite subgroup $G \subset \mathrm{Aut}(X)$, form the quotient, together with quotient map

$$q : X \longrightarrow X/G = Y.$$

If X is projective, this actually exists as a projective scheme.

Note: this does not hold true for proper schemes.

Examples

Example: Let X be a curve. The function field $F = k(X)$ can be written as finite extension of $k(t)$.

Any Galois group $G \subset \mathrm{Aut}(F/k)$ extends to $\mathrm{Aut}(X)$. Quotient $Y = X/G$ has function field $k(Y) = F^G$.

Specialize further: Suppose $X \subset \mathbb{P}^2$ is an elliptic curve, for Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

In coordinates, the sign involution is given by

$$(x, y) \longmapsto (x, -(y + a_1x + a_3))$$

This indeed yields $X/\{\pm 1\} = \mathbb{P}^1$.

Second method: invertible sheaves

Second method: Let $\mathscr{L}$ be an invertible sheaf on X. Suppose it is globally generated, with with $V = H^0(X, \mathscr{L})$ of dimension $n+1$.

Gives unique morphism

$$r : X \longrightarrow \mathbb{P}^n \quad \text{with } \mathscr{L} = r^*(\mathscr{O}_{\mathbb{P}^n}(1)).$$

Factors over image $Y \subset \mathbb{P}^n$, comes with Stein factorization

$$X \longrightarrow Y' \longrightarrow Y \subset \mathbb{P}^n.$$

Consequently, in algebraic geometry a lot of effort goes into understanding invertible sheaves and their global sections.

Example: elliptic curves

Example: Suppose X is an elliptic curve. Let $\mathscr{L}$ be invertible of degree $d \geq 1$.

If $d = 1$ then $h^1(\mathscr{L}) = h^0(\mathscr{L}^\vee) = 0$ by Serre Duality, thus

$$h^0(\mathscr{L}) = \chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_X) = 1$$

from Riemann–Roch. So $\mathscr{L}$ is not globally generated, only have $X \dashrightarrow \mathbb{P}^0$.

If $d = 2$ then $h^0(\mathscr{L}) = 2$. Now $\mathscr{L}$ is globally generated, get double covering $r : X \to \mathbb{P}^1$.

For $d = 3$ we get $h^0(\mathscr{L}) = 3$, and $r : X \to \mathbb{P}^2$ becomes an embedding.

Picard scheme

Given proper scheme X, it is very desirable to know all possible invertible sheaves $\mathscr{L}$, say up to isomorphism.

Turns out that isomorphism classes $[\mathscr{L}]$ can be seen as points on another scheme, the **Picard scheme** $\mathrm{Pic}_{X/k}$, cum grano salis. But how to define and construct it?

Grothendieck's insight: Regard $\mathrm{Pic}_{X/k}$ as **something** that trivially exists. Then prove that this something has the **property** of being a scheme.

The functor

As a scheme, $\text{Pic}_{X/k}$ would be describable by polynomial equations, and therefore it makes sense to speak of R-**valued solutions**.

These solution sets should be

$$\text{Pic}_{X/k}(R) = \text{Pic}(X \otimes R),$$

where R runs through all k-algebras. This is functorial in R.

So we regard $\text{Pic}_{X/k}$ as a **contravariant functor** on (Aff/k); have to prove that it is **representable by a scheme**. By Yoneda, this indeed defines the desired scheme.

Counterexample

But it fails: The functor $R \mapsto \mathrm{Pic}(X \otimes R)$ is not representable!

Counterexample: Consider the quadric curve

$$X : \quad T_0^2 + T_1^2 + T_2^2 = 0$$

in $\mathbb{P}^2$ over the field $k = \mathbb{R}$. This is a Brauer–Severi curve. Have $\mathrm{Pic}(X \otimes \mathbb{C}) = \mathbb{Z}$, with canonical element $[\mathscr{O}(1)]$.

If $R \mapsto \mathrm{Pic}(X \otimes R)$ would be describable by polynomials equations over $k = \mathbb{R}$, the canonical complex solution $[\mathscr{O}(1)]$ must be Galois invariant. Therefore produces a real solution, contradiction!

Sheafification

All in vain? No, one has to analyse the problem!

The counterexample exploits that the contravariant functor $R \mapsto \mathrm{Pic}(X \otimes R)$ does not satsify the sheaf axiom.

So let's replace the presheaf by its sheafification...

Representability

Theorem. (Grothendieck, Murre, Artin) The sheafification of $R \mapsto \mathrm{Pic}(X \otimes R)$ is representable by a group scheme $\mathrm{Pic}_{X/k}$. Its connected component $\mathrm{Pic}^0_{X/k}$ is of finite type, and the quotient $\mathrm{NS}_{X/k}$ is étale, with finitely generated stalk.

Alexander Grothendieck (1928–2014)

Jakob Murre (*1929)

Michael Artin (*1934)

But what did it become?

So we have something representable, but we are not sure anymore about what it represents precisely... What happens upon sheafification?

Recall that for each continuous map $f : Y \to Z$ of toplogical spaces and each abelian sheaf F on Y, the higher direct images $R^i f_*(F)$ are the sheafification of $V \mapsto H^i(f^{-1}(V), F)$.

We have $\mathrm{Pic}(X \otimes R) = H^1(X \otimes R, \mathbb{G}_m)$. Let $f : X \to \mathrm{Spec}(k)$ be the structure morphism.

Idea: Reinterpret as continuous functor $f : (\mathrm{Aff}/X) \to (\mathrm{Aff}/k)$, so sheafification of above gives first direct image $R^1 f_*(\mathbb{G}_m)$.

Leray–Serre spectral sequence

Now recall that continuous maps $f : Y \to Z$ come with the **Leray–Serre spectral sequence**

$$E_2^{rs} = H^r(Z, R^s f_*(F)) \Longrightarrow H^{r+s}(Y, F).$$

On the left is the E_2-page, on the right the abutment.

From this we get the **five-term exact sequence**:

$$0 \to H^1(Z, f_*F) \to H^1(Y, F) \to H^0(Z, R^1 f_*F) \to H^2(Z, f_*F) \to H^2(Y, F)$$

The five-term sequence

Apply this to the structure morphism $f : X \to \mathrm{Spec}(k)$ and the sheaf $F = \mathbb{G}_m$. Get exact sequence:

$$0 \to H^1(k, \mathbb{G}_m) \to H^1(X, \mathbb{G}_m) \to H^0(k, R^1 f_* \mathbb{G}_m) \to H^2(k, \mathbb{G}_m) \to H^2(X, \mathbb{G}_m)$$

Interpret first and second cohomology as Picard and Brauer group; above yields

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X).$$

The term in the middle is the **group of rational points on the Picard scheme**!

Obstructions

From this exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X)$$

we see:

Theorem. *For a rational point $l \in \mathrm{Pic}_{X/k}$, the* obstruction *to come from an invertible sheaf $\mathscr{L}$ lies in the Brauer group* $\mathrm{Br}(k)$.

Some corollaries

Corollary 1. *If the Brauer group* $\mathrm{Br}(k)$ *vanishes, we have an identification* $\mathrm{Pic}(X) = \mathrm{Pic}_{X/k}(k)$.

The induced mapping $f^* : \mathrm{Br}(k) \to \mathrm{Br}(X)$ admits a retraction, provided that $f : X \to \mathrm{Spec}(k)$ has a section. Thus:

Corollary 2. *If* X *contains a rational point, we have an identification* $\mathrm{Pic}(X) = \mathrm{Pic}_{X/k}(k)$.

In any case, Brauer groups for fields are torsion. This gives:

Corollary 3. *For a rational point* $l \in \mathrm{Pic}_{X/k}$, *some positive multiple* ml *comes from an invertible sheaf* $\mathscr{M}$.

Othter moduli spaces

Similar principals hold for many **other moduli problems**:

Instead of invertible sheaves $\mathscr{L}$, one may consider locally free sheaves $\mathscr{E}$ of fixed rank $r \geq 0$.

To get representable functors, one has to restrict to sheaves without undue or excessive automorphisms.

Depending on context, one restricts attention to sheaves that are simple/stable/semi-stable...

Poincaré sheaves

Leads to the moduli space $M_{X,r,\xi}$ of simple sheaves $\mathscr{E}$, with fixed rank r. Also likes to fix determinant $\xi \in \mathrm{Pic}(X)$.

But there is always scalar multiplication, giving $\mathrm{Aut}(\mathscr{E}) = k^\times$. So we **never** have a fine moduli space—there **cannot be a universal object** $\mathscr{P}$ on $M_{X,r,d} \times X$.

Though the universal $\mathscr{P}$ does not exist, its projectivization

$$\mathbb{P}(\mathscr{P}) = \mathrm{Proj}(\mathrm{Sym}(\mathscr{P}))$$

does, because there scalar multiplications become identities! This is a family of Brauer–Severi varieties over $M_{X,r,\xi}$.

Poincaré classes

From the moduli problem, we thus get a canonical element

$$[\mathbb{P}(\mathscr{P})] \in \mathrm{Br}(M_{X,r,d}).$$

Like to call it **Poincaré class**.

Theorem. (Balaji, Biswas, Gabber, Nagaraj) *If X is a smooth projective curve, the group* $\mathrm{Br}(M_{X,r,\xi})$ *is generated by the Poincaré class.*

Theorem. (Reineke, S) *Similar results hold for moduli spaces $M_{Q,d}$ of representations of certain quivers Q with dimension vector d.*

Thank you very much for the attention!